DIFFERENTIAL GEOMETRY 88-826 HOMEWORK SET 3

Due Date: 11 may '22

1. Recall that the m-th exterior power $\bigwedge^m(\mathbb{R}^m)$ of $\mathbb{R}^m$ is spanned by the single element $\omega = e_1 \wedge e_2 \wedge \cdots \wedge e_m$. Consider the 2-multivector

$$\alpha = e_1 \wedge e_2 + e_3 \wedge e_4 + \cdots + e_{2n-1} \wedge e_{2n} \in \bigwedge^2(\mathbb{R}^{2n}).$$

Express the product $\alpha \wedge \alpha \wedge \cdots \wedge \alpha$ (n times) explicitly as a multiple of $\omega \in \mathbb{R}^{2n}$.

2. Let M be a n-dimensional Riemannian manifold. Consider a coordinate chart (A, u) in M. Let f be a smooth function on A and consider the differential 2-form $\eta = f(u^1, \ldots, u^n)du \wedge dv$ in A, where du and dv are among the coordinate forms du^i. Prove that the 4-form $dd\eta$ identically vanishes.

3. Consider the Eisenstein lattice $L_E \subseteq \mathbb{C}$ spanned by the cube roots of unity. Let L_E^* be its dual lattice. Calculate the product $\lambda_1(L_E^*)\lambda_1(L_E)$.