88826 Differential geom., moed A, 27 jul '14

Duration of the exam: 3 hours.

All answers must be justified by providing complete proofs.

1. Consider a parametrized surface $x(u^1, u^2)$ in $\mathbb{R}^3$.
   (a) Define the mean curvature H of the surface and determine whether it is possible to express H in terms of the coefficients g_{ij} of the metric and their partial derivatives of suitable orders, and if so provide such an expression with proof.
   (b) Determine whether it is possible to express the coefficent Γ_{ij}^k in terms of the g_{ij} and their partial derivatives of suitable orders, and if so provide such an expression with proof.
   (c) Determine whether it is possible to express the quantity $L_{i[j}L^k_{\ell]}$ in terms of the g_{ij} and their partial derivatives of suitable orders, and if so provide such an expression with proof.

2. Let $M \subset \mathbb{R}^3$ be a compact convex surface without boundary. Let $G : M \to S^2$ be the Gauss map sending each point of M to the unit normal vector at the point. Let (θ, φ) be spherical coordinates on S^2. Let $x = \theta \circ G$ and $y = \varphi \circ G$ be the corresponding coordinates on M. Let $g_{ij}(x, y)$ be the coefficients of the metric on M with respect to coordinates (x, y) and let $K = K(x, y)$ be the Gaussian curvature of M.
   (a) Evaluate the integral $\int_{x=0}^{x=2\pi} \int_{y=0}^{y=\pi} K(x, y)\sqrt{\det g_{ij}}\,dxdy$, with a detailed proof.
   (b) Calculate the Gaussian curvature of the metric $\frac{1}{(x-y)^2}(dx^2+dy^2)$ whenever $x \neq y$.

3. The following expressions use the Enstein summation convention. Simplify as much as possible:
   (a) $\langle x_{\ell j}, n_k\rangle \left(\delta^k{}_m\right) g^{m\ell}$.
   (b) $\langle x_j, x_{pq}\rangle \left(\delta^j{}_r\right)$.
   (c) $\langle x_{pqr}, x_m\rangle$.
   (d) $\delta^a{}_b\, g_{ca}\, g^{bd}\, \delta^c{}_d$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a real function.
   (a) State the definition of microcontinuity of a function at a point and express the property of continuity of f on $\mathbb{R}$ in terms of microcontinuity.
   (b) Express the property of uniform continuity of f on $\mathbb{R}$ in terms of microcontinuity.

(c) Given a continuous function f on $[0,1]$, define a hyperfinite partition and use it to prove the extreme value theorem for f (see also part (d) below).

(d) Explain which notion of continuity was used in the proof in part (c) and precisely in what way.

5. Let $\mathbb{C}$ be the field of complex numbers. Let $A \subset \mathbb{C}$ be the subfield consisting of all points of the form $a + ib$ where $a, b \in \mathbb{Q}$.

(a) Give a detailed definition of the structure $\mathbb{N}^*$ of hypernatural numbers in terms of a nonprincipal ultrafilter $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$, determine whether $\mathbb{N}^*$ contains nonzero infinitesimals, and if so provide an example.

(b) Give a detailed definition of the field $\mathbb{Q}^*$ of hyperrational numbers in terms of a nonprincipal ultrafilter, determine whether $\mathbb{Q}^*$ contains nonzero infinitesimals, and if so provide an example.

(c) Let $A^* = \{a + ib : \ a, b \in \mathbb{Q}^*\}$. Let $I = \{a + ib \in A^* : \ \mathrm{st}(a) = 0, \ \mathrm{st}(b) = 0\}$. Let $B = \{a + ib \in A^* : \ a \text{ is finite}, \ b \text{ is finite}\}$. Determine whether the quotient B/I is isomorphic to either of the structures $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, A$ or their natural extensions $\mathbb{N}^*$, $\mathbb{Z}^*, \mathbb{Q}^*, \mathbb{R}^*$, $\mathbb{C}^*$, A^*, with proof.

GOOD LUCK!