88826 Differential geom., moed A, 5 aug '15

Duration of the exam: 3 hours.

All answers must be justified by providing complete proofs.

1. Let X be the infinitesimal generator of a flow $\theta = \theta(t,p)$ on a manifold M. Give a detailed proof of the fact that X is invariant under θ, defining all the relevant concepts.

2. Let F a prevector field on a manifold. Give a detailed proof of the fact that F is invariant under the hyperreal flow defined by F, defining all the relevant concepts.

3. Let $\mathcal{F}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and ${}^*\mathbb{R}$ the corresponding hyperreal line.

   (a) Present a detailed definition of ${}^*\mathbb{R}$ in terms of $\mathcal{F}$.
   (b) Consider a sequence $\mathcal{A} = \langle A_n \subset \mathbb{R} : n \in \mathbb{N}\rangle$ of subsets of $\mathbb{R}$. Give a detailed definition of the internal subset $[\mathcal{A}] \subset {}^*\mathbb{R}$ and specify when a hyperreal $u = [u_n] \in {}^*\mathbb{R}$ belongs to $[\mathcal{A}]$.
   (c) Consider the sequence $\langle 1, 2, 4, 8 \ldots \rangle$ and let H be the corresponding hyperreal. Determine whether the set $[0, H] = \{x \in {}^*\mathbb{R} : 0 \le x \le H\}$ is internal, and if so describe it by a sequence of sets as in part (b).
   (d) Let $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$. Determine whether the set $\{x \in {}^*\mathbb{R} : 0 \le x \le e\}$ is internal, and if so describe it by a sequence of sets as in part (b).

4. If c is an upper bound for a set $A \subset \mathbb{R}$ we will write $A \le c$. The completeness property of $\mathbb{R}$ asserts that if A is bounded from above, then there is a least upper bound $d \in \mathbb{R}$ for A, or in formulas

$$(\forall A \subset \mathbb{R})\big[(\exists c \in \mathbb{R})[A \le c] \Rightarrow (\exists d \in \mathbb{R})[A \le d] \wedge (\forall e \in \mathbb{R})[A \le e \Rightarrow d \le e]\big]$$

   (a) Express the condition $A \le c$ by an explicit first-order formula with quantification only over numbers.
   (b) Reformulate the completeness property given by the formula above in a way amenable to an application of the transfer principle.
   (c) Apply the transfer principle to the resulting formula so as to obtain a correct statement over ${}^*\mathbb{R}$.
   (d) Give an example of the failure of the naive application of transfer to the formula above.

GOOD LUCK!