88-826 Differential Geometry, moed A
Bar Ilan University, Prof. Katz
Date: 3 july '19
Duration of the exam: 3 hours
Each of 5 problems is worth 20 points; bonus problem is 10 points
All answers must be justified by providing complete proofs

1. Let $X = \mathbb{C}^{n+1} \setminus \{0\}$ be the collection of $(n+1)$-tuples $x = (x^0, \ldots, x^n)$ distinct from the origin. Define an equivalence relation $\sim$ between $x, y \in X$ by setting $x \sim y$ if and only if there is a complex number $t \neq 0$ such that $y = tx$, i.e.,

$$y^i = tx^i, \quad i = 0, \ldots, n.$$

Denote by $[x]$ the equivalence class of $x \in X$. Define the complex projective space, $\mathbb{CP}^n$, as the collection of equivalence classes $[x]$, i.e., $\mathbb{CP}^n = \{[x] \colon x \in X\}$. Prove that $\mathbb{CP}^n$ is a smooth manifold and determine its real dimension.

2. For each of the lattices $L_n \subseteq \mathbb{C}$, find the conformal parameter $\tau(\mathbb{C}/L_n)$:
   (1) L_1 spanned by the roots of the polynomial $z^3 - 1$;
   (2) L_2 spanned by 2 and i;
   (3) L_3 spanned by 1 and $3 + i$.

3. This problem concerns the exterior differential complex on a manifold M.
   (1) Define the term $\Omega^k(M)$ of the complex.
   (2) Define the differentials d_1 and d_2 in the following segment of the exterior differential complex: $\Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \Omega^3(M)$.
   (3) Prove that the segment is exact, i.e., $d_2 \circ d_1(\xi) = 0$ for all 1-forms $\xi \in \Omega^1(M)$.

4. Let $C \in \mathbb{R}$. Compute the Gaussian curvature of the metric $f^2(dx^2 + dy^2)$ with conformal factor $f(x, y) = \frac{1}{1+C(x^2+y^2)}$.

5. Let $\mathbb{T}^n$ be the n-dimensional torus.
   (1) Compute the de Rham cohomology group $H^0_{dR}(\mathbb{T}^n)$.
   (2) Let $S^1 = \mathbb{T}^1$ be the circle. Compute the de Rham cohomology group $H^1_{dR}(S^1)$.

6. (**bonus**) Let $\mathbb{R}/\mathbb{Z}$ denote the circle of length 1. Consider the cylinder $C_H = \mathbb{R}/\mathbb{Z} \times [0, H]$ of height $H > 0$, with coordinates $x \in \mathbb{R}/\mathbb{Z}$ and $y \in [0, H]$. Suppose a surface M contains an annulus conformally equivalent to C_H. Find the best upper bound for the ratio $\frac{\mathrm{sys}_1^2(M)}{\mathrm{area}(M)}$.

Good luck!