88-826 Differential Geometry, moed A
Bar Ilan University, Prof. Katz
Date: 13 july '20
Duration of the exam: 3 hours
Each of 5 problems is worth 20 points; bonus problem is 10 points
All answers must be justified by providing complete explanations and proofs

1. In $\mathbb{R}^3$ with standard basis (e_1, e_2, e_3), consider the unit sphere $S^2 \subseteq \mathbb{R}^3$. Construct an atlas for the manifold S^2 consisting of two coordinate charts, (A, u) and (B, v) as follows.
   (a) Let $A = S^2 \setminus \{e_3\}$. Given a point $p \in A$, consider the line $\ell_x^+ \subseteq \mathbb{R}^3$ through p and e_3. Let $u\colon A \to \mathbb{R}^2$ map each point $p \in A$ to the intersection of the line ℓ_p with the (x, y)-plane equipped with polar coordinates (r, θ). Find an explicit formula for u.
   (b) Let $B = S^2 \setminus \{-e_3\}$ and $p \in B$. Consider the line $\ell_p^- \subseteq \mathbb{R}^3$ through p and $-e_3$. Let $v\colon B \to \mathbb{R}^2$ map each point $p \in B$ to the intersection of the line ℓ_p^- with the (x, y)-plane equipped with polar coordinates (r', θ'). Find an explicit formula for v.
   (c) Determine the transition function for the overlap $A \cap B$.
   (d) Find the metric coefficients of the unit sphere metric with respect to the coordinate u defined in part (a).

2. This question deals with orientations on manifolds.
   (a) Let M be an oriented manifold with boundary. Give a detailed definition of the notion of the induced orientation on the boundary ∂M.
   (b) Let $b > 0$ and let D be the unbounded region $D = \{(x, y) \in \mathbb{R}^2 \colon x^2+y^2 \geq b^2\}$ endowed with the standard orientation $dx\wedge dy$. Calculate the induced orientation on ∂D and compare it to $d\theta$.
   (c) Let $b > 0$ and let D be the unbounded region $D = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 \geq b^2\}$ endowed with the standard orientation $dx \wedge dy \wedge dz$. Calculate the induced orientation on ∂D and compare it to the orientation defined by $\alpha_{FS} = \sin\phi\, d\theta \wedge d\phi$.

3. For each of the lattices $L_n \subseteq \mathbb{C}$, find the conformal parameter $\tau(\mathbb{C}/L_n)$:
   (a) L_1 spanned by the roots of the polynomial $z^3 - 8$;
   (b) L_2 spanned by 2 and i;
   (c) L_3 spanned by 1 and $3 + i$.

4. This problem deals with de Rham cohomology.
   (a) Compute (with proof) all of the de Rham cohomology groups $H^k_{dR}(\mathbb{R}/\mathbb{Z})$.

(b) Let $L \subseteq \mathbb{C}$ be the Gaussian integers. Compute (with proof) the de Rham cohomology group $H^2_{dR}(\mathbb{C}/L)$.

5. Let M be an closed connected orientable 8-dimensional manifold. Assume that $b_2(M) = 1$ and that a class $\omega \in H^2_{dR}(M)$ satisfies $\omega^{\cup 4} \neq 0$.

(a) Give a detailed definition of what it means for a de Rham class $\omega \in H^2_{dR}(M)$ to be an integer class.

(b) Consider a metric g on M. Give detailed definitions of the norm $\| \ \|$ in $\Lambda^2(T_pM)$; the norm $\| \ \|_\infty$ in $\Omega^2 M$; and the norm $\| \ \|^*$ in de Rham cohomology.

(c) Let $\eta \in \omega$ be a representative differential 2-form. Estimate the integral $\int_M \eta \wedge \eta \wedge \eta \wedge \eta$ in terms of the comass of η as well as the total volume $\mathrm{vol}(M)$ of M.

(d) Provide (with proof) the best upper bound for the following ratio: $\mathrm{stsys}_2(g)^4/\mathrm{vol}(g)$.

6. (**bonus**) Consider the cylinder $C_H = \mathbb{R}/\mathbb{Z} \times [0, H]$ of height $H > 0$, with coordinates $x \in \mathbb{R}/\mathbb{Z}$ and $y \in [0, H]$. Suppose a surface M contains an annulus conformally equivalent to C_H, where $\mathbb{R}/\mathbb{Z}$ is noncontractible in M. Determine the best upper bound for the ratio $\frac{\mathrm{sys}_1^2(M)}{\mathrm{area}(M)}$.

Good luck!