88-826 DIFFERENTIAL GEOMETRY, MOED B, 19 SEPT '08

Duration of the exam: $2\frac{1}{2}$ hours.

All answers must be justified by providing complete proofs.

1. Consider the polar coordinates (r, θ) of a point p in the Euclidean plane.

(a) Find a natural orthonormal basis, in terms of the polar coordinates, for the cotangent plane T_p^* at p when p is not the origin.
(b) Find a natural orthonormal basis, in terms of the polar coordinates, for the tangent plane T_p when p is not the origin.
(c) Consider the cotangent line $T_p^* S^1$ at a point p of the circle of radius $r_0 > 0$. Consider the lattice $L_0 \subset T_p^*$ spanned by the 1-form $d\theta$. Calculate $\lambda_1(L_0)$.
(d) Consider the tangent line T_p at a point p of the circle of radius $r_0 > 0$. Consider the lattice $L_1 \subset T_p$ spanned by $\frac{\partial}{\partial \theta}$. Calculate $\lambda_1(L_1)$.
(e) Determine whether or not the differential form $d\theta$ on S^1 is a coboundary, i.e. lies in the image of the map $C^\infty(S^1) \to \Omega^1(S^1)$ defined by the exterior derivative.

2. Given a metric g on a torus $\mathbb{T}^2$, let $\lambda_1(g)$ be the length of a shortest noncontractible loop (lul'ah bilti-kvitzah) $\gamma_0 \subset \mathbb{T}^2$.

(a) Let $a, b > 0$, and consider the 2-parameter family $g_{a,b}$ of tori of revolution in 3-space (with circular section) obtained by rotating the circle

$$(x - a)^2 + y^2 = b^2.$$

(b) Write down an explicit formula for $\lambda_1(g_{a,b})$ in terms of the parameters a, b, with proof.
(c) Define the first homology group $H_1(\mathbb{T}^2)$.
(d) Let λ_2 the least length of a noncontractible loop whose homology class is not proportional to that of the loop γ_0 as above. Write down an explicit formula for λ_2 in terms of the parameters, with proof.

3. Continuing with the notation of the previous problem, consider the ratio

$$SR_{1,2}(g) = \frac{\lambda_1 \lambda_2}{area(g)}.$$

(a) Determine the range of the ratio $SR_{1,2}(g_{a,b})$.
(b) Determine if the endpoints of the range are attained.
(c) if an endpoint is attained, describe the metrics attaining it.
(d) if an endpoint is not attained, describe a sequence of metrics whose ratio tends to the endpoint.

4. This problem is concerned with flat tori (not necessarily imbedded in Euclidean space) and their invariants λ_1 and λ_2 as in the previous problems.

(a) Give the definition of a flat torus, and describe a parametrisation of the family of flat tori.
(b) Write down an explicit formula for λ_1 and λ_2 in terms of the parameters, with proof.
(c) Find the range of the ratio $SR_{1,2}(g)$, and determine if the endpoints of the range are attained.
(d) If an endpoint is attained, describe the metrics attaining it; if an endpoint is not attained, describe a sequence of metrics whose ratio tends to the endpoint.

5. For each of the following lattices L, find L^* and compute $\lambda_1(L^*)$, after presenting the definition in part (a):

(a) Define the notion of the dual lattice in Euclidean n-space.
(b) The lattice $L_G \subset \mathbb{C}$ spanned over $\mathbb{Z}$ by the roots of $z^4 = 81$.
(c) Let $a, b, c > 0$ such that $a \leq b \leq c$. The lattice $L_{a,b,c} \subset \mathbb{R}^3$ is spanned by ae_1, be_2, and ce_3.
(d) The lattice $L_E \subset \mathbb{C}$ spanned by the roots of $z^6 = 64$.

GOOD LUCK!

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