28 aug 2007. Projective Geometry, 88-524-01. moed Bet. Final Exam

ALL ANSWERS MUST BE JUSTIFIED

1. Consider the field $F = F_{11}$ with 11 elements. Let A be the affine plane over F, let FP^1 be the projective line over F, and let FP^2 be the projective plane over F.

(a) Find the number of points and the number of lines in FP^1;
(b) Find the number of points and the number of lines in A;
(c) Find the number of points and the number of lines in FP^2;
(d) Calculate the number of points in the intersection between the pair of projective lines in FP^2 defined by the equations $2x + y + 3z = 0$ and $3x + 4y + 2z = 0$ in homogeneous coordinates;
(e) Calculate the number of points in the intersection between the pair of projective lines in FP^2 defined by the equations $x - 4y + 3z = 0$ and $3x - y - 2z = 0$ in homogeneous coordinates.

2. Let A, B, C be points on a line ℓ, and P point not on ℓ.

(a) Give a precise definition of a harmonic 4-tuple.
(b) Describe a geometric construction of a point D such that (A, B, C, D) is harmonic.
(c) Draw a sequence of at least three careful and precise drawings illustrating each step of the construction.
(d) Describe the construction dual to the one in (a), starting with a triple of lines a, b, c concurrent in point L, and line p not through L.

3. Let $R(A, B, C, D)$ be the cross-ratio (yachas hakaful) of points on the real line, when $A = \infty$, $B = 0$, and $C = 1$. Let $D_k = \frac{2k-5}{3}$, where $k = 0, 1, 2, 3$.

(a) What are the possible values of the cross ratio when $k = 0$?
(b) Let $f(k)$ be the total number of distinct values of the cross-ratio of all the permutations of the 4-tuple (A, B, C, D_k). Calculate $f(k)$ as an explicit function of the index $k = 0, \ldots, 3$.

4. Projective transformations of the completed real line $\mathbb{R} \cup \infty$ have the following form:

$$f(x) = \frac{ax + b}{cx + d},$$

where $ad - bc = 1$. Find a projective transformation $y = f(x)$ which sends

(a) the points $x = 1, 0, \infty$ respectively to the points $y = 3, -4, \infty$;
(b) the points $x = 1, 0, \infty$ respectively to the points $y = 4, 1, 0$.

5. Let $\mathcal{C}$ be a circle. A triangle is called self-dual (duali le'atzmo) if every vertex (kodkod) is polar (with respect to $\mathcal{C}$) to the opposite side (tsela mimul). Let ABC be a self-dual triangle.

(a) Prove that the center of the circle is the intersection point of the altitudes (govahot) of the triangle ABC.
(b) Prove that one of the vertices of ABC is necessarily inside the circle, and two vertices are outside.
(c) Present a careful drawing to illustrate items (a) and (b).

GOOD LUCK!