# Percolation in Interdependent and Interconnected Networks: Abrupt Change from Second to First Order Transition

Yanqing Hu[1,2,3], Baruch Ksherim[2], Reuven Cohen[4], Shlomo Havlin[2]
*1.School of Mathematics Southwest Jiaotong University, Chengdu 610031, China.*
*2. Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel*
*3. Department of Systems Science, School of Management and Center for Complexity Research, Beijing Normal University, Beijing 100875, China*
*4. Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel*
(Dated: December 22, 2011)

Robustness of two coupled networks system has been studied only for dependency coupling (S. Buldyrev et. al., Nature, 2010) and only for connectivity coupling (E. A. Leicht and R. M. D'Souza, arxiv:09070894). Here we study, using a percolation approach, a more realistic coupled networks system where both interdependent and interconnected links exist. We find a rich and unusual phase transition phenomena including hybrid transition of mixed first and second order i.e., discontinuities like a first order transition of the giant component followed by a continuous decrease to zero like a second order transition. Moreover, we find unusual discontinuous changes from second order to first order transition as a function of the dependency coupling between the two networks.

During the last decade complex networks have been studied intensively, where most of the research was devoted to analyzing the structure and functionality isolated systems modeled as single non-interacting networks [1–11]. However, most real networks are not isolated, as they either complement other networks ("interconnected networks"), must consume resources supplied by other networks ("interdependent networks") or both [12–16]. Thus, real networks continuously interact one with each other, composing large complex systems, and with the enhanced development of technology, the coupling between many networks becomes more and more significant.

Two different types of coupled networks models have been studied. Buldyrev et. al. [17] investigated the robustness of coupled systems with only interdependence links. In these systems, when a node of one network fails, its dependent counterpart node in the other network also fails. They found that this interdependence makes the system significantly more *vulnerable* [17, 18]. In the same time, Leicht and D'Souza [19] studied the case where only connectivity links couple the networks, i.e., "interconnected networks", and found that the interconnected links make the system significantly more *robust*. However, real coupled networks often contain both types of links, interdependent as well as interconnected links. For example, the airport and the railway networks in Europe are two coupled networks composing a transportation system. In order to arrive to an airport, one usually uses the railway. Also, people arriving to the country by airport usually use the railway. In this system, if the airport is disabled by some strike or accident, the passengers can still use the nearby railway station and travel to their destination or to another airport by train, so the two networks are coupled by connectivity links. On the other hand, if the railway network is disabled, the airport traffic is damaged, and if the airport is disabled, the railway traffic is damaged, so both networks are coupled by dependency links as well. The important characteristics of such systems, is that a failure of nodes in one network carries implications not only for this network, but also on the function of other dependent networks. In this way it is possible to have cascading failures between the coupled networks, that may lead to a catastrophic collapse of the whole system. Nevertheless, small clusters disconnected from the giant component in one network can still function through interconnected links connecting them to the giant component of other network. Thus, the inter-connectivity links *increase* the robustness of the system, while the inter-dependency links *decrease* its robustness. Here we study the competition of the two types of inter-links on robustness using a percolation approach, and find unusual types of phase transitions.

Let us consider a system of two networks, $A$ and $B$, which are coupled by both dependency and connectivity links. The two networks are partially coupled by dependency links, so that a fraction $q_A$ of $A$-nodes depends on nodes in network $B$, and a fraction $q_B$ of $B$-nodes depends on the nodes in network $A$, with the following two exceptions: a node from one network depends on no more than one node from the other network, and assuming that node $A_i$ depends on node $B_j$, then if $B_j$ depends on some $A_h$, then $h = i$ (see Fig. 1). In addition, the connectivity links within each network and between the networks (see Fig. 1) can be described by a set of degree distributions $\{\rho^A_{k_A,k_{AB}}, \rho^B_{k_B,k_{BA}}\}$, where $\rho^A_{k_A,k_{AB}}$ ($\rho^B_{k_B,k_{BA}}$) denotes the probability of an $A$-node ($B$-node) to have $k_A$ ($k_B$) links to other $A$-nodes ($B$-node) and $k_{AB}$ ($k_{BA}$) links towards $B$-nodes ($A$-nodes). In this manner we get a two dimensional generating function describing all the connectivity links [19], $\mathcal{G}^A_0(x_A, x_B) = \sum_{k_A,k_{AB}} \rho^A_{k_A,k_{AB}} x_A^{k_A} x_B^{k_{AB}}$, and $\mathcal{G}^B_0(x_A, x_B) = \sum_{k_B,k_{BA}} \rho^B_{k_B,k_{BA}} x_A^{k_{BA}} x_B^{k_B}$.

The cascading process is initiated by randomly removing a fraction $1 - p$ of the $A$-nodes and all their connectivity links. Because of the interdependence between the networks, the nodes in network $B$ that depend on the removed $A$-nodes are also removed along with their connectivity links. As nodes and links are removed, each network breaks up into connected components (clusters). We assume that when the network is fragmented, the nodes belonging to the largest component (giant component) connecting a finite fraction of the network are still functional, while nodes that are parts of the remaining smaller clusters become dysfunctional, unless there exist a

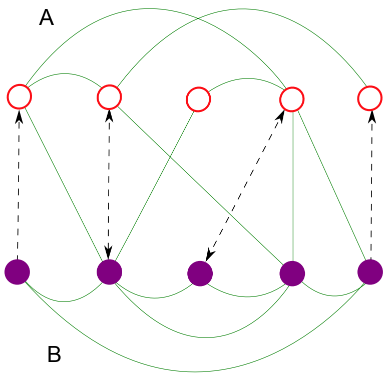


FIG. 1: (Color online) Two types of inter-links where the dependency links (dashed arrows) are not necessarily bidirectional. The nodes of A and B are randomly connected with *connectivity-links* (full line). The functionality of some of the A nodes (red open circle) *depend* on B-nodes (purple solid circle) and vice versa.

path of connectivity-links connecting these small clusters to the largest component of the other network. Since the networks have different topologies, the removal of nodes and related dependency links, is not symmetric in both networks, so that, a cascading process occurs, until the system either becomes fragmented or stabilizes with a giant component.

Let $g_A(\varphi, \phi)$ and $g_B(\varphi, \phi)$ be the fraction of A-nodes and B-nodes in the giant components after the percolation process initiated by removing a fraction of $1-\varphi$ and $1-\phi$ of networks A and B respectively [11]. The functions $g_A(\varphi, \phi)$ and $g_B(\varphi, \phi)$ depend only on $\mathcal{G}_0^A(x_A, x_B)$ and $\mathcal{G}_0^B(x_A, x_B)$ (For details see Appendix) and the cascading process can be described by the following set of equations,

$$\begin{aligned}
&\varphi_1 = p, \quad \phi_1 = 1, && P_1^A = \varphi_1 g_A(\varphi_1, \phi_1), \qquad (1)\\
&\phi_2 = 1 - q_B\big(1 - p g_A(\varphi_1, \phi_1)\big), && P_2^B = \phi_2 g_B(\varphi_1, \phi_2),\\
&\varphi_2 = p\big(1 - q_A\,(1 - g_B(\varphi_1, \phi_2))\big), && P_2^A = \varphi_2 g_A(\varphi_2, \phi_2),\\
&\phi_3 = 1 - q_B\big(1 - p g_A(\varphi_2, \phi_2)\big), && P_3^B = \phi_3 g_B(\varphi_2, \phi_3),
\end{aligned}$$

where, $\phi_i, \varphi_i$ are the remaining fraction of nodes at stage $i$ of the cascade of failures and $P_i^A$, $P_i^B$ are the corresponding giant components of networks $A$ and $B$, respectively. Generally, the $n^{\underline{th}}$ step is given by the equations,

$$\begin{aligned}
&\varphi_n = p\big(1 - q_A\,(1 - g_B(\varphi_{n-1}, \phi_n))\big), \qquad (2)\\
&\phi_n = 1 - q_B\big(1 - p g_A(\varphi_{n-1}, \phi_{n-1})\big),\\
&P_n^A = \varphi_n g_A(\varphi_n, \phi_n), \quad P_n^B = \phi_n g_B(\varphi_{n-1}, \phi_n).
\end{aligned}$$

By introducing two new notations

$$u_A = g_A(\varphi_\infty, \phi_\infty), \quad u_B = g_B(\varphi_\infty), \phi_\infty, \qquad (3)$$

we can write the equations at the end of the cascading process,

$$\phi_\infty = p\big(1 - q_A(1 - u_B)\big), \quad \varphi_\infty = 1 - q_B(1 - p u_A), \qquad (4)$$

and the giant components are,

$$\begin{aligned}
P_\infty^A &= u_A \phi_\infty = u_A p\big(1 - q_A(1 - u_B)\big), \qquad (5)\\
P_\infty^B &= u_B \varphi_\infty = u_B\big(1 - q_B(1 - p u_A)\big).
\end{aligned}$$

In the case where all degree distributions of intra- and inter-links are *Poisson* distributed, the functions obtain a simple form. Assume $\bar{k}_A$ and $\bar{k}_B$ are the average intra-links degrees in networks $A$ and $B$, and $\bar{k}_{AB}$, $\bar{k}_{BA}$ are the average inter-links degrees between $A$ and $B$ (allowing the case $\bar{k}_{AB} \neq \bar{k}_{BA}$, since the two networks may be of different sizes), we obtain,

$$\begin{aligned}
u_A &= 1 - e^{-\bar{k}_A p u_A\left(1 - q_A(1 - u_B)\right) - \bar{k}_{AB} u_B\left(1 - q_B(1 - p u_A)\right)}, \qquad (6)\\
u_B &= 1 - e^{-\bar{k}_{BA} p u_A\left(1 - q_A(1 - u_B)\right) - \bar{k}_B u_B\left(1 - q_B(1 - p u_A)\right)}.
\end{aligned}$$

Generally, for fixed parameters $\bar{k}_A, \bar{k}_B, \bar{k}_{AB}, \bar{k}_{BA}, q_A, q_B$ and $p$, it is often impossible to achieve an explicit formula for the giant components $P_\infty^A$ and $P_\infty^B$. However, one can still solve Eqs. (6) graphically and substitute the numerical solution to Eqs. (5). For example, we study the case where $\bar{k}_A = \bar{k}_B \equiv \bar{k}$ and $\bar{k}_{AB} = \bar{k}_{AB} \equiv \bar{K}$. Fig. 2**a** compares the numerical with the simulation results for $P_\infty^A$ and $P_\infty^B$ as a function of $p$, showing that the analytical results of Eqs. (5) and (6) are in excellent agreement with the simulations.

Next we are interested in the properties of the phase transition under random attack, so first we determine the conditions when transition does not occur. This is the case when even all nodes of network A are removed ($p = 0$), for a given $q_B < 1$, there still exists a giant component in network B (see circles in Fig. 2**a**) and no phase transition occurs. For Poisson degree distributions, if after the removal of all B-nodes that depend on the attacked A-nodes, the new average intra-link degree in network B is less than one, i.e.,

$$\bar{k}_B(1 - q_B) < 1, \qquad (7)$$

a phase transition occurs. Therefore, the following analysis is based on condition (7). In addition, we always set both dependency strengths, $q_A$ and $q_B$, to be larger than zero.

When the phase transition is of second order, i.e., the giant components at the percolation threshold is zero. Thus, according to the limit of system (6) at $u_A = u_B = 0$ we obtain the second order threshold, for $q_A \neq 1$,

$$p_c^{II} = \frac{1 - \bar{k}_B(1 - q_B)}{\big(\bar{k}_A + (\bar{k}_{BA}\bar{k}_{AB} - k_A k_B)(1 - q_B)\big)(1 - q_A)}. \qquad (8)$$

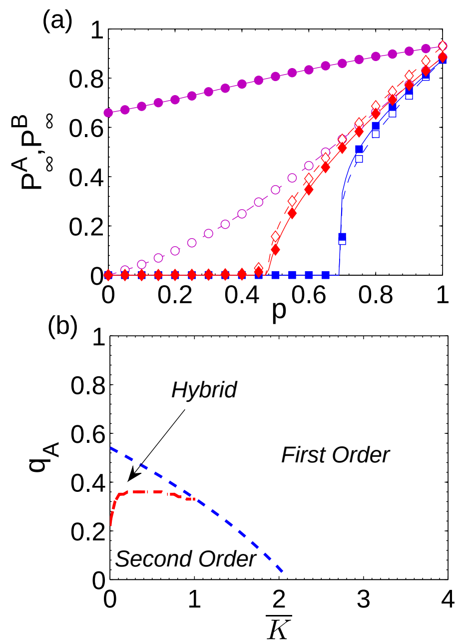


FIG. 2: (Color online) **a.** Giant components $P^A_\infty$ and $P^B_\infty$ vs. fraction of remaining nodes, $p$, for $N = 10000$, $\bar{k} = 2$ and $\overline{K} = 1$. Networks $A$ (open symbols) and $B$ (full symbols) for different $(q_A, q_B)$ pairs: $(0.8, 0.1)$ (○); $(0.8, 0.8)$ (◇); $(0.1, 0.1)$ (□). The symbols represent simulations and the lines the theory. We see three types of behaviors: no phase transition (○), second order phase transition (◇) and first order phase transition (□). **b.** Phase diagram showing the first order, second order and hybrid phase transition regimes and the boundaries, for $q_B = 1, \bar{k} = 3$. In the second order transition regime, between the two dashed curve (red and blue) is the hybrid phase transition regime (details in Fig. 3**c** and in the SI). Since the hybrid transition is continuous in the neighborhood of $p_c$, and jump occurs well above $p_c$ we classify a hybrid phase transition as a second order phase transition.

When $q_A = 1$ and $0 \le q_B < 1$ this threshold becomes

$$p_c^{II} = \frac{1}{\bar{k}_B(1-q_B)} > 1,$$

which together with Eq. (7) implies that the phase transition must be of first order at $p_c^I < 1$ that will be determined later.

Solving the first equation of system (6), yields an explicit formula for $u_B$, so that system (6) can be rewritten as

$$u_B = -\frac{\log(1-u_A) + k_A p(1-q_A)u_A}{k_A p q_A u_A + k_{AB}[1-q_B(1-pu_A)]} \equiv H_1(u_A), \quad (9)$$
$$u_B = 1 - e^{-\bar{k}_{BA}u_A p\left(1-q_A(1-u_B)\right)-\bar{k}_B u_B\left(1-q_B(1-u_A p)\right)} \equiv H_2(u_A).$$

and the intersection of the two curves (maximum solution of $u_A, u_B$) is the solution of the system. When the phase transition is first order and $p = p_c^I$, the curves of Eqs. (9) are tangentially touching at the solution point, where,

$$\left(\frac{dH_1}{du_A} = \frac{dH_2}{du_A}\right)\Big|_{p=p_c^I}. \quad (10)$$

Obviously, $u_A$, $u_B$ and $p$ can be treated as variables of Eqs. (9) and (10). Solving these equations, the minimal solution of $p$ and the corresponding maximum $u_A$, $u_B$ of the minimal $p$ is the solution of the system at criticality.

When networks A and B are fully dependent, i.e., $q_A = q_B = 1$, system (6) yields a simple form

$$u_A = 1 - \exp\left\{-pu_A u_B\left(\bar{k}_A + \bar{k}_{AB}\right)\right\},$$
$$u_B = 1 - \exp\left\{-pu_A u_B\left(\bar{k}_B + \bar{k}_{BA}\right)\right\}.$$

The size of the mutual giant component, $P_\infty$, is thus given by,

$$P_\infty = P^A_\infty = P^B_\infty = p\left(1 - e^{-P_\infty(\bar{k}_A+\bar{k}_{AB})}\right)\left(1 - e^{-P_\infty(\bar{k}_B+\bar{k}_{BA})}\right), \quad (11)$$

which is similar to the solution of fully interdependent system [17], where the only difference is that the degrees of networks A and B are now replaced by $\bar{k}_A + \bar{k}_{AB}$ and $\bar{k}_B + \bar{k}_{BA}$, respectively. Thus, interestingly, in a fully interdependent coupled networks adding connectivity inter-links has the same effect as increasing the intra-degree of the corresponding networks and therefore, in this case, the phase transition must be of first order. From Eqs. (9) and (10), one can get the threshold,

$$p_c^I = \frac{1}{k_A(1-u_A)\left[-1+(1-u_A)^\alpha - u_A\alpha(1-u_A)^{\alpha-1}\right]}, \quad (12)$$

where, $\alpha \equiv (\bar{k}_B+\bar{k}_{BA})/(\bar{k}_A+\bar{k}_{AB})$, and $u_A$ satisfies the equation,

$$u_A = 1 - \exp\left\{\frac{u_A[1-(1-u_A)^\alpha]}{(1-u_A)[-1+(1-u_A)^\alpha - u_A\alpha(1-u_A)^{\alpha-1}]}\right\}. \quad (13)$$

For fully interdependent system, both networks are of the same size and therefore $\bar{k}_{AB} = \bar{k}_{BA}$.

By substituting $p_c^{II}$ from Eq. (8) into Eqs. (9) and (10) and evaluating both $u_A$ and $u_B$ we can derive and draw in the phase diagram, the boundary between the first and second order transitions (see dashed line in Fig. 2**b**). The most interesting phenomenon, which to the best of our knowledge, has not been observed before, is that when the phase transition changes from first to second, there are discontinuities (abrupt jumps) of $P^A_\infty(p_c), P^B_\infty(p_c)$ in the phase transition boundary (see Fig. 3**a**). On the boundary between first and second order phase transition, $p_c^I = p_c^{II}$. Substituting $p_c^I$ with $p_c^{II}$ in Eq. (9) and evaluating both $u_A$ and $u_B$ we can obtain the boundary between the first and second order transitions. When we reduce the three equations to one equation, $u_A$, $u_B$ should always be the maximum non-negative solution in [0,1]. When Eq. (9) and (10) has more than one solution, we always choose the minimal non-negative value, $p_c^{min}$ and the corresponding maximum value solution $u_A^{max}$, $u_B^{max}$ as the physical solution at the threshold. In some regime of the boundary, $u_A^{max} > 0$ and $u_B^{max} > 0$, and of course $p_c^{min}, u_A = 0$, $u_B = 0$ also is the system solution. It means that there exist two intersections and both of them satisfy the tangential condition (as shown in Fig. 4) on the boundary. This implies that when the order of the phase transition changes from first to second, $P^A_\infty\left(p_c\right), P^B_\infty\left(p_c\right)$ are discontinuous. This phenomenon contrasts most systems possessing both first and second order transitions. In physical systems usually, the first order jump in the order parameter, and

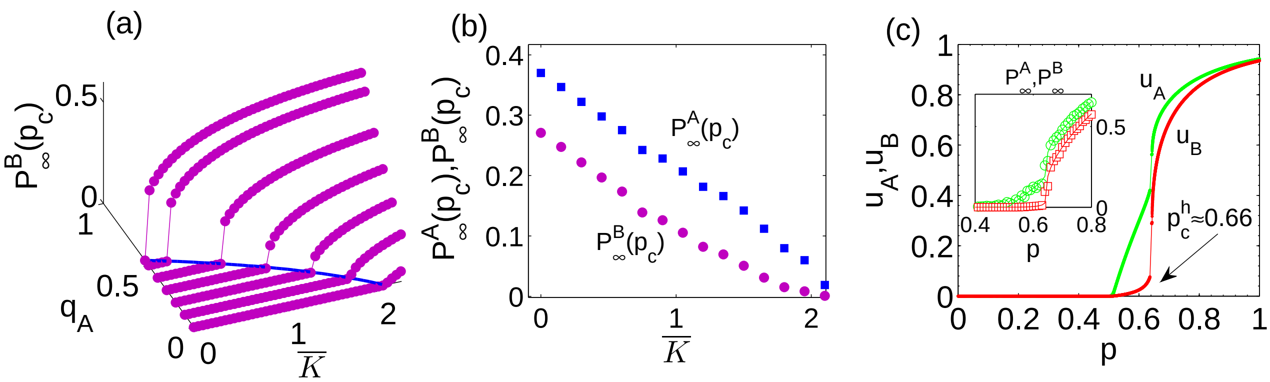


FIG. 3: **a.** Size of giant components vs. dependency and connectivity links strength, for $q_B = 1$ and $\overline{k} = 3$. The giant components size at $p_c$ changes from zero to a finite value while changing $q_A$ and $\overline{K}$. When $q_A$ and $\overline{K}$ are at the boundary of different phase transitions, the jump occurs. **b.** The values of $P^A_\infty(p_c)$ (○), $P^B_\infty(p_c)$ (□) along the boundary for $q_B = 1$ and $\overline{k} = 3$. **c.** Hybrid phase transition, for $q_B = 1, q_A = 0.35$, $\overline{k} = 3$ and $\overline{K} = 0.1$. According to Eqs. (5), $P^A_\infty$ and $P^B_\infty$ have the same properties as $u_A$ and $u_B$ respectively. At $p \approx 0.66$ the values of $u_A$ and $u_B$ jump, and then for lower $p$ valve continuously approach zero. In the inset, simulation and theoretical results are symbols and lines respectively.

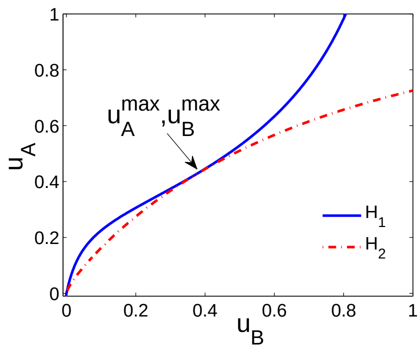


FIG. 4: Abrupt jump on the boundary, here $q_A = 0.394, q_B = 0.8, \overline{k} = 3, \overline{K} = 0.2$. $p^I_c = p^{II}_c = 0.5464$ which is the threshold of the system. Although, both intersections (one of which is at the origin) satisfy the tangential condition, the $u^{max}_A, u^{max}_B$ is the physical solution and the transition is of the first order.

related properties, such as the specific heat, present a continuous change along the transition line when the system changes from first to second order.

In addition to the existence of jumps in $P^A_\infty(p_c), P^B_\infty(p_c)$ at the boundary between the first and second order phase transitions, we find another unusual phenomenon. When one network strongly depends on the other, there exist hybrid phase transitions. By hybrid phase transition we mean that when increasing the attack strength, $1-p$, the size of the giant component jumps at $p^h_c$ from a large value to a small value, and then continuously decreases to zero. A similar behavior has been found in bootstrap percolation [20]. Since the second order transition is characterized by a giant component which is continuous in the neighborhood of $p_c$, we regard, the hybrid phase transition regime as a second order phase transition regime (see Fig. 2**b**). For the hybrid phase transition, there exists a threshold $p^h_c$ at which the jump occurs. For $p$ just below $p^h_c$, the solution of Eqs. (9) for $u_A$, $u_B$, will jump to lower values . After the jump, when $p$ is further decreased, $u_A$ and $u_B$ approach to zero continuously which implies that the giant components sizes change to zero continuously (see Fig. 3**c**).

For the three equations system ( (9) and (10)) , the minimum solution of $p^{min}$ in [0,1] is the $p_c$ (physical solution). Besides $p^{min}$, if Eq. (9) has another solution $p^h_c \in (0,1)$ and corresponding solution $u^h_A$, $u^h_B$, we can find the hybrid phase transition. $(p^h, u^h_A, u^h_B)$ means that when $p$ is little less than $p^h_c$, the solution $u_A$, $u_B$ of the first two equations of Eq. (9) will jump to small values. After the jump, when we continue to decrease $p$ to $p_c = p^{min}$, $u_A$, $u_B$ will move to 0 continually. For example, for the parameters $q_A = 0.35, q_B = 1, \overline{k} = 3$ and $\overline{K} = 0.1$, we obtain $p_c = 0.556$ and $p^h_c = 0.66$. When $p$ is just below 0.66, the giant components drops to smaller positive values like in a first order phase transition. After this discontinuous drop, the giant component's size continuously decreases to zero when decreasing $p$ from 0.66 to 0.556 like a second order phase transition (see Fig. 5).

In summary, we studied the cascade of failures in coupled networks, when both interdependent and interconnected links exist, using a percolation approach. Although our detailed analysis is for ER networks, the theory can be applied to any network systems topology. We find that the existence of inter-connectivity links between interdependent networks, introduces rich and intriguing phenomena through the process of cascading failures. Increasing the strength of interconnecting links can change the transition behavior significantly and often brings up some counterintuitive phenomenon, such as changing the transition from second order to first order (as seen in Fig. 2**b**). We also find an unusual abrupt jump in the boundary between first and second order phase transitions at the critical point, which, to the best of our knowledge, has not been observed earlier in physical systems. Moreover, when one of the networks strongly depends on the other network, unusual hybrid phase transitions are observed.

We thank Amir Bashan for helpful discussions. This work is partially supported by ONR, DFG, DTRA, EU project Epi-

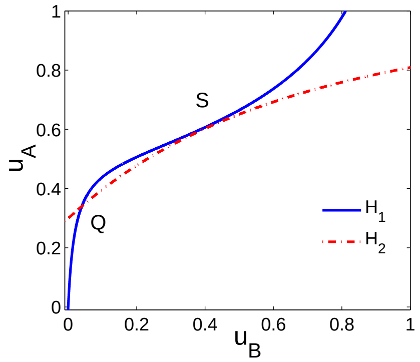


FIG. 5: Hybrid transition analysis, for $q_B = 1, q_A = 0.35, \overline{k} = 3$ and $\overline{K} = 0.1$, here $p_c \approx 0.556$. $p^h \approx 0.66$. The maximum intersection S satisfies tangential condition. When continuously decreasing $p$, the solution of the system jumps from the maximum intersection S to the minimum intersection Q and then continuously decrease to zero.

work and the Israel Science Foundation for financial support. Y. Hu is supported by NSFC under Grant No. 60974084, 60534080. R. Cohen thanks the BSF for partial support.

[1] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[2] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[3] R. Albert and A.-L.Barabási, Rev. Mod. Phys. **74**, 47(2002).
[4] Y. Hu *et al.*, Phys. Rev. Lett. **106**, 108701 (2011).
[5] C. Song *et al.*, Nature (London) **433**, 392 (2005); Nature Physics **2**, 275 (2006).
[6] R. Pastor-Satorras and A. Vespignani, Evolution and Structure of the Internet: A Statistical Physics Approach (Cambridge University Press, 2006).
[7] S. N. Dorogovtsev and J. F. F. Mendes, Evolution of Networks: from Biological nets to the Internet and WWW (Oxford University Press, New York, 2003).
[8] A. Barrat *et al.*, Dynamical processes on complex networks (Cambridge University Press, England, 2008).
[9] D. S. Callaway *et al.*, Phys. Rev. Lett. **85**, 5468 (2000).
[10] R. Cohen *et al.*, Phys. Rev. Lett. **85**, 4626 (2000); **86**, 3682 (2001).
[11] M. E. J. Newman, Networks: An Introduction. (Oxford University press 2010).
[12] V. Rosato *et al.*, Int. J. Crit. Infrastruct. 4, 63-78, (2008).
[13] N. K. Svendsen and S. D. Wolthusen, Information Security Technical Report, **21**, page 44-55, (2007).
[14] S. Rinaldi *et al.*, IEEE Contr. Syst. Mag. 21, 11C25 (2001).
[15] J. Peerenboom *et al.*, in Proc. CRIS/DRM/IIIT/NSF Workshop Mitigat. Vulnerab. Crit. Infrastruct. Catastr. Failures (2001).
[16] J. C. Laprie *et al.*, SAFECOMP-2007, **4680**, 54 (2007).
[17] S. Buldyrev *et al.*, Nature, **464**, 7291 (2010),
[18] R. Parshani *et al.*, Phys. Rev. Lett. **105**, 048701 (2010).
[19] E. A. Leicht, R. M. D'Souza, arxiv:0907.0894.
[20] G. J. Baxter, *et al.*, Phys. Rev. E **82**, 011103 (2010).