

# Dynamic networks and directed percolation

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**Abstract.** - We introduce a model for dynamic networks, where the links or the strengths of the links change over time. We solve the model by mapping dynamic networks to the problem of directed percolation, where the direction corresponds to the time evolution of the network. We show that the dynamic network undergoes a percolation phase transition at a critical concentration  $p_c$ , that decreases with the rate  $r$  at which the network links are changed. The behavior near criticality is universal and independent of  $r$ . We find that for dynamic random networks fundamental laws are changed. (i) The size of the giant component at criticality scales with the network size  $N$  for all values of  $r$ , rather than as  $N^{2/3}$  in static networks. (ii) In the presence of a broad distribution of disorder, the optimal path length between two nodes in a dynamic network scales as  $N^{1/2}$ , compared to  $N^{1/3}$  in a static network.

Network theory has answered many questions concerning static networks [1–14], but many real networks are dynamic in the sense that their links, or the strengths of their links, change with time. For example, in social networks friendships are formed and dissolved, while in communication networks, such as the Internet, the load (weight) on the links changes continually. Models for dynamic networks have been studied in the context of epidemic models in biology [15], as well as for routing and gossiping algorithms in computer science [16, 17]. In social networks, dynamic models such as the reciprocity model and the actor oriented model [18–21], include rate and objective functions that allow to control and optimize the changes in the network.

In this manuscript we focus on the general physical aspects of dynamic networks. Fundamental questions that have been extensively studied in static networks are still open for dynamic networks. Here we ask: (i) Does the dynamic network undergo a percolation phase transition, above which order  $N$  of the network nodes are still connected and below which the network breaks into small clusters? (ii) If so, what is the critical concentration of links for which the transition occurs, and how does it depend on the dynamics? (iii) What are the properties near criticality?

We start with a definition for a percolation process. In a percolation process on a network each link can be tra-

versed with some probability  $p$ . Under this restriction, all the nodes that can be reached by a walker located on an initial node, are regarded as one cluster. At the critical percolation threshold  $p = p_c$  a second order percolation transition occurs. For  $p > p_c$ , a spanning cluster exists, and thus, a walker traversing the network is able to reach an order  $N$  of the network nodes. For  $p < p_c$  the network collapses into small clusters and the walker is trapped inside a small cluster [5, 6, 22–24]. Directed percolation (DP) is a special kind of percolation in which the walker is limited to advance only in a specific direction that is defined to be the longitudinal axis [26, 27].

Consider an  $N$ -node network with  $M$  links where each link has a weight  $w$  chosen from a given distribution. A unit time step is defined as the time required for a walker to traverse a single link. At the end of each time step the links are rewired with probability  $r$  (Fig. 1(a)). Without loss of generality [25], we assume that a walker traversing the network must advance to a new node at the beginning of each time step. If at some stage he is not able to do so he is removed from the network. Now, even if there is no path between nodes  $A$  and  $B$  at a specific time, a walker traversing the network may be able to pass from point  $A$  to point  $B$  because new links continuously appearing. Likewise, even if a path between  $A$  and  $B$  exists at a given time it may be disconnected before a walker is able to traverse it (Fig. 1(a)). Moreover, even if a path between

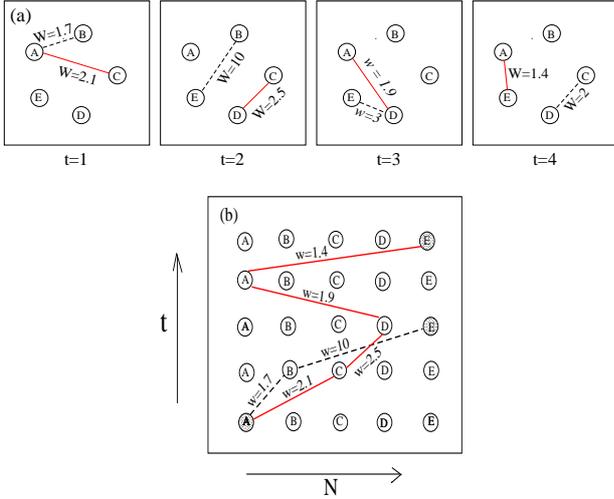


Fig. 1: (a) (a) A five-node dynamic network at different time steps. The solid (red) links compose the shortest path between A and E (see (b)). (b) The y axis corresponds to the time dimension and each successive layer of two rows corresponds to a network configuration at a different time step (as shown in (a)) The solid (red) line is the optimal path (sum of its weights is minimal) between node A and node E, even though a shorter path exists, shown as a (black) dashed line. Note, that not all links from (a) are presented, only the links leading from A to E.

two nodes always exists, the optimal path between the two nodes (the path along which the sum of the weights is minimal) may change. Fig. 1(b) demonstrates a scenario where the path between node A and E after four steps is the optimal path, rather than the shorter path between the nodes after two steps.

In a percolation process on a dynamic network, after the links are rewired on each step, each link is set to be traversable with probability  $p$ . We now argue that percolation on a dynamic network is equivalent to the problem of directed percolation (DP) in infinite dimensions [26, 27]. To show this, the time evolution of the network is represented by adding another axis, which corresponds to a time axis. In this extended representation, every two successive rows along the time axis represent a layer that corresponds to a network configuration at a different time step [Fig. 1(b)]. As a result, each node in the original network is represented as a set of nodes (one for each time step) in the extended representation. In a percolation process on such a network representation a walker is restricted to advance only in one direction - along the time axis [Figs. 1(a) and 1(b)] and therefore the percolation process is actually a directed percolation where the time  $t$  is equivalent to step  $t$  in directed percolation. This correspondence between dynamic networks and DP not only gives a meaning to DP in networks but more importantly allows us to apply the results known from DP at criticality to dynamic networks.

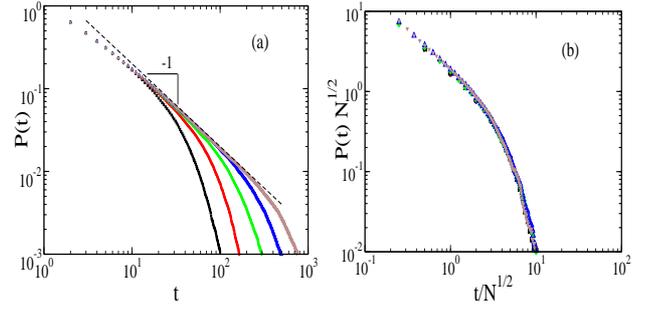


Fig. 2: (Color online) (a) Simulation results for the survivability  $P(t)$  and its cutoff, at criticality, for dynamic networks of different network sizes. From left to right:  $N = 100, 400, 1600, 6400, 25600$ . (b) The data collapse shown demonstrates that  $P(t)$  in dynamic networks is universal when  $t$  is scaled by  $N^{1/2}$ .

Networks can be regarded as infinite dimensional structures since no spatial constraints exist. Therefore since our mapping is exact, we expect that at  $p_c$  the critical properties of dynamic networks be the same as DP in infinite dimensions. The relevant critical properties for DP are [26, 27]:  $S(t)$ , the giant component size, scales as  $S(t) \sim t^2$  and  $P(t)$ , the survivability (the probability of reaching layer  $t$  when growing a cluster), scales as  $P(t) \sim t^{-1}$ . Figures 2(a)-(b) and 3(a) present simulation results showing that for dynamic networks we observe these scaling relations.

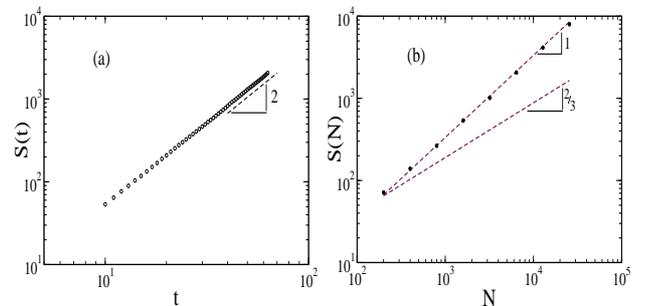


Fig. 3: (a) The cluster size,  $S(t)$ , at criticality scales as  $t^2$ . (b) Simulation results (dots) supporting the relation  $S(N) \sim N$  (Eq. 3) (upper dashed line), the cluster size in dynamic networks, compared to the known  $S(N) \sim N^{2/3}$  (bottom dashed line), the cluster size in static networks.

To learn about the size-dependent properties of dynamic networks we determine the DP properties as a function of the network size  $N$ , rather than as a function of  $t$ . In DP at criticality, the infinite dimensional relationship between  $w$ , the width in the transverse axes, and  $t$ , the length in the longitudinal axes, is  $w \sim t^{1/2}$ . The upper critical dimension  $d_c$  is the lowest dimension for which the system has the DP properties of an infinite dimensional system. For DP this value is  $d_c = 4 + 1$  (1 corresponds to the longitudinal axis), so the relation between the system size

at the upper critical dimension and the size of a dynamic network is given by  $N \sim w^4$  (the power 4 comes from the 4 transverse dimensions of  $d_c$ ). Since  $w \sim t^{1/2}$  we conclude that:

$$t \sim N^{1/2}. \quad (1)$$

Therefore for a dynamic network of size  $N$  at criticality, the survivability  $P(t)$  decays exponentially after a time  $t_\times$  with  $t_\times \sim N^{1/2}$  [28], as shown in Figure 2(b).

Figure 2(a) shows that for different values of  $N$  and  $t < t_\times \sim N^{1/2}$ ,  $P(t) \sim t^{-1}$ , as expected from DP in infinite dimensions. The exponential decay for  $t > t_\times$  can also be seen, in agreement with Eq. (1). Fig. 2(b) shows the collapse of survivability data after scaling by  $N^{1/2}$ , supporting again Eq. (1).

The size of the giant component  $S(N)$ , at criticality, is derived by substituting Eq. (1) in the DP relation  $S(t) \sim t^2$ .

$$S(N) \sim N. \quad (2)$$

Figure 3(b) presents simulation results for dynamic networks supporting this scaling relationship, compared to the known relationship for static networks, where  $S(N)$  scale as  $S(N) \sim N^{2/3}$  [22, 23]. The two systems clearly have different behavior and properties at criticality, and thus belong to two different universality classes. Fig. 4(b) and Fig. 4(d) present simulation results for  $S(N)$  at different values of  $p$  indicating that the dynamic network undergoes a phase transition at some critical value of  $p$ .

We find that the behavior of dynamic networks at criticality is universal and independent of the rate  $r$  in which the links are rewired (Fig. 4(c)). However, the critical concentration,  $p_c$ , for which the percolation phase transition occurs depends on  $r$  [Fig. 4(a)].

The dependence of  $p_c$  on  $r$  can be explained as follows: For simplicity assume that instead of rewiring a rate  $r$  of the links on each step, all the links are rewired on a percentage  $r$  of the steps and no links are rewired on  $1 - r$  of the steps. Consider a node  $i$  reached by following a link. If there has been a rewiring after traversing this link then the node now has  $k$  links to neighbors, chosen randomly from the degree distribution. Each of these links can be traversed with probability  $p$ , and therefore the average branching factor [29] is  $\langle kp \rangle = p\langle k \rangle$ .

Now suppose there has been no rewiring after traversing a link. In ER networks, each node has on average  $p\langle k \rangle$  outgoing neighbors as before in addition to the link through which it arrived that points to its parent (see note [30]). Links to the parent node, however, exist for all nodes reached from the same parent (siblings). The number of neighbors,  $x$ , of the parent, excluding node  $i$ , is Poisson distributed with mean  $\langle k \rangle$ . Assuming node  $i$  has already reached its parent the other  $k$  siblings each have a probability  $p$  to be reached and then a probability  $p$  to return. Thus, the number of neighbors,  $x$ , that can return to the parent is binomially distributed as

$$P(x) = \binom{k}{x} p^{2x} (1 - p^2)^{k-x}.$$

When calculating the branching factor at each step we should count the parent node only once. The contribution of the link back to the parent to the branching factor of each of the parents sibling is therefore inverse to the total number of siblings. The average contribution of each sibling with degree  $k$  is

$$\left\langle \frac{1}{x+1} \right\rangle_k = \sum_{x=0}^k \binom{k}{x} \frac{p^{2x} (1 - p^2)^{k-x}}{x+1} = \frac{1 - (1 - p^2)^{k+1}}{(k+1)p^2}.$$

Using the fact that  $x$  is Poisson distributed, the average contribution is

$$\begin{aligned} \left\langle \frac{1}{x+1} \right\rangle &= \sum_{k=0}^{\infty} \left\langle \frac{1}{x+1} \right\rangle_k \frac{e^{-\langle k \rangle} \langle k \rangle^k}{k!} \\ &= \frac{1 - e^{-p^2 \langle k \rangle}}{p^2 \langle k \rangle} \equiv f(p). \end{aligned}$$

After  $n$  steps, we have on average  $nr$  steps with rewiring and  $n(1 - r)$  steps without rewiring. Thus, the total branching factor is

$$(p\langle k \rangle)^{nr} (p\langle k \rangle + f(p))^{n(1-r)}.$$

For the process to be at criticality, this factor should be 1, leading to

$$p_c \langle k \rangle g(p_c) = 1, \quad (3)$$

where

$$g(p_c) = [1 + f(p_c)/\langle k \rangle]^{1-r}. \quad (4)$$

Fig. 4(a) compares simulation results with the numerical solution of Eq. (3).

The correspondence of dynamic networks to DP can also predict the general scaling of the optimal path in a dynamic network with a broad distribution of disorder. In the limit of a broad distribution of disorder, Ref. [31] has shown that, at criticality, the optimal path exists mainly along the giant cluster. Therefore for static networks the optimal path length scales with the average distance between nodes on the percolation cluster:  $\ell_{\text{opt}} \sim N^{1/3}$  [31]. In our dynamic network model the average distance between nodes on the percolation cluster scales as  $\langle \ell \rangle \sim N^{1/2}$  (the average path length is dominated by the typical length [Eq. (1)]), suggesting that in dynamic ER networks the optimal path scales as

$$\ell_{\text{opt}} \sim N^{1/2}. \quad (5)$$

Figure 5 shows simulation results for the scaling of the optimal path length with  $N$  in a dynamic network compared to a static network. The results for dynamic networks are in full agreement with Eq. (5).

What makes the results in a dynamic network so different from the static case? The difference lies in the larger number of available configurations. While in static networks the percolation cluster is composed from paths built from  $N$  network nodes, in dynamic networks the network

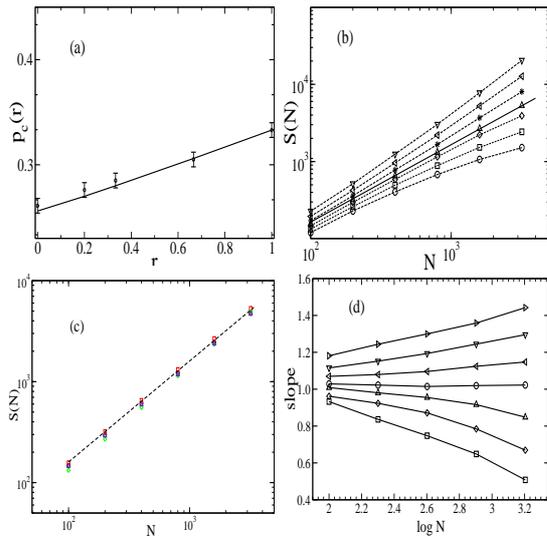


Fig. 4: (a) Simulation fit to the formula of  $p_c$  for a network with  $\langle k \rangle = 3$ , as a function of the rate  $r$  at which the links are rewired. The value of  $p_c$  was calculated by finding the value of  $p$  for which  $S(N)$  gives a straight line. (b) Simulation results for  $S(N)$  in a network with  $\langle k \rangle = 3$  and  $r = 2/3$  are presented for different values of  $p$  showing that the network undergoes a phase transition at  $p_c$ . (c) The value of  $S(N)$  at  $p_c$  for several different values of  $r$  that all collapse to one line. In all cases the slope equals 1 as predicted by the critical exponent for dynamic networks. (d) The slope between every two consecutive points in (b). The slope decreases below  $p_c$ , increases above  $p_c$  and is a constant equal to 1 at  $p_c$ .

is represented by  $N' = N^{3/2}$  nodes [32]. The evolution of the network over time generates many more possible configurations, enabling the percolation cluster to become much larger. Substituting  $N$  with  $N'$  in the known percolation cluster formula for static networks,  $S(N) \sim N^{2/3}$ , yields  $S(N') \sim S(N^{3/2}) \sim N$ , which further confirms our results for  $S(N)$  in dynamic networks.

The same explanation is true also for the optimal path, where a longer and more optimal path is available due to the increased number of available configurations. For example, an optimal path reaching some node  $A$  may find it optimal not to advance to the near neighbor  $B$  at the next step but rather to first visit  $C$  and only then come back to  $B$  since at the later time node  $B$  is more optimally connected to the destination node.

Representing a dynamic network as a directed network [Fig. 2(a)] composed of  $N' = N^{3/2}$  nodes allows the “same” node to be counted more than once in the percolation cluster, therefore raising the question if the distinct number of nodes on the percolation cluster also scales with  $N$ . To determine the number of different nodes of the original network in a component of size  $M$  on the directed network, consider the following argument: The links between consecutive layers of the directed network are chosen randomly. Therefore, each link leads to a random node in

the original network independently and with uniform distribution. After  $\lambda$  links have been followed, assume that  $D_\lambda$  nodes have already been visited. The probability to reach a new node by following the next link, is therefore  $1 - D_\lambda/N$ . Thus,  $D_{\lambda+1} = D_\lambda$  with probability  $D_\lambda/N$ , and  $D_{\lambda+1} = D_\lambda + 1$  with probability  $1 - D_\lambda/N$ . The expected number of distinct nodes  $E(D)$  reached after  $\lambda$  links have been followed from the starting node is therefore  $E(D_{\lambda+1}) = E(D_{\lambda-1}) + E(1 - D_{\lambda-1}/N)$ . This reduces to  $E(D_{\lambda+1}) = 1 + (1 - 1/N)E(D_\lambda)$  which indicates that for large  $\lambda$

$$\frac{E(D_M)}{N} = 1 - \left(1 - \frac{1}{N}\right)^M \approx 1 - e^{-M/N}. \quad (6)$$

Thus, when the size of a component in the directed network is of order  $N$  a finite fraction of the visited nodes are new and the size of the induced component on the original network is also of order  $N$ .

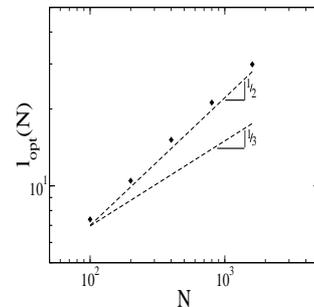


Fig. 5: The optimal path for strong disorder scales as  $\ell_{opt}(N) \sim N^{1/2}$  in dynamic networks compared to  $\ell_{opt}(N) \sim N^{1/3}$  in static networks.

In summary, we introduced a model for dynamic networks which was solved by mapping the model to directed percolation in 4+1 dimensions. The DP longitudinal axis is mapped to the time axis along which the dynamic network evolves. We showed that dynamic networks exhibit different properties and critical exponents near criticality. Therefore they belong to a different universality class than static networks. While in static networks,  $S(N)$ , the size of the giant component at criticality, scales as  $S(N) \sim N^{2/3}$ , in dynamic networks  $S(N) \sim N$ . Even though the properties of dynamic networks are universal and independent of the rate  $r$  at which the links are changed, the critical concentration,  $p_c$ , for which the phase transition occurs depends on  $r$ . We also showed that the optimal path in dynamic networks scales as  $\ell_{opt} \sim N^{1/2}$ , compared to  $\ell_{opt} \sim N^{1/3}$  in static networks.

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