Optimal Path in Random Networks with Disorder: A mini review

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Abstract

We review the analysis of the length of the optimal path $\ell_{\rm opt}$ in random networks with disorder (i.e. random weights on the links). In the case of strong disorder, in which the maximal weight along the path dominates the sum, we find that $\ell_{\rm opt}$ increases dramatically compared to the known small world result for the minimum distance $\ell_{\rm min}$: for Erdős-Rényi (ER) networks $\ell_{\rm opt} \sim N^{1/3}$, while for scale free (SF) networks, with degree distribution $P(k) \sim k^{-\lambda}$, we find that $\ell_{\rm opt}$ scales as $N^{(\lambda-3)/(\lambda-1)}$ for $3 < \lambda < 4$ and as $N^{1/3}$ for $\lambda \geq 4$. Thus, for these networks, the small-world nature is destroyed. For $2 < \lambda < 3$, our numerical results suggest that $\ell_{\rm opt}$ scales as $\ln^{\lambda-1} N$. We also find numerically that for weak disorder $\ell_{\rm opt} \sim \ln N$ for ER models as well as for SF networks. We also study the transition between the strong and weak disorder regimes in the scaling properties of the average optimal path $\ell_{\rm opt}$ in ER and SF networks.

1 Introduction

Recently much attention has been focused on the topic of complex networks which characterize many biological, social, and communication systems [1–3]. The networks are represented by nodes associated to individuals, organizations, or computers and by links representing their interactions. The classical model for random networks is the Erdős-Rényi (ER) model [4–6]. An important quantity characterizing networks is the average distance (minimal

hopping) $\ell_{\min}$ between two nodes in the network of total N nodes. For the Erdős-Rényi network $\ell_{\min}$ scales as $\ln N$ [6], which leads to the concept of "small worlds" or "six degrees of separation". For scale-free (SF) [1] networks $\ell_{\min}$ scales as $\ln\ln N$, this leads to the concept of ultra small worlds [7].

In most studies, all links in the network are regarded as identical and thus a crucial parameter for information flow including efficient routing, searching, and transport is $\ell_{\min}$. In practice, however, the weights (e.g., the quality or cost) of links are usually not equal, and thus the length of the optimal path ℓ_{opt}, minimizing the sum of weights, is usually longer than the minimal hopcount distance $\ell_{\min}$. For example, the cost could be the time required to transit the link. There are often many traffic routes from point A to point B with a set of time delays τ_i, associated with each link along the path. The fastest (optimal) path is the one for which $\sum_i \tau_i$ is a minimum, and often the optimal path has more links than the shortest path. In many cases, the selection of the path is controlled by most of the weights (e.g., total cost) contributing to the sum. This case corresponds to weak disorder (WD). However, in other cases, for example when the distribution of disorder is very broad a *single* weight dominates the sum. This situation—in which one link controls the selection of the path—is called the strong disorder limit (SD). An example for SD is when a transmission at a constant high rate is needed (e.g., in broadcasting video records over the Internet). In this case the narrowest band link in the path between the transmitter and receiver controls the rate of transmission. This limit is also called the "ultrametric" limit and we refer to the optimal path in this limit as the min-max path.

2 Algorithms

2.1 Construction of the Networks

To construct an ER network of size N with average node degree $\langle k\rangle$, we start with $\langle k\rangle N/2$ edges and randomly pick a pair of nodes from the total possible $N(N-1)/2$ pairs to connect with an edge. The only condition we impose is that there cannot be multiple edges between two nodes. When $\langle k\rangle > 1$ almost all nodes of the network will be connected with high probability.

To generate scale-free (SF) graphs of size N, we employ the Molloy-Reed algorithm [8] in which each node is first assigned a random integer k from a power law distribution $P(k > \bar{k}) = (\bar{k}/k_0)^{-\lambda+1}$, where k_0 is the minimal number of links for each node. Next we randomly select a node and try to connect each of its k links with randomly selected k nodes that still have free positions for links. When $k_0 > 1$ there is a high probability that the network

is fully connected.

The strong disorder limit can be implemented by assigning to each link a potential barrier ϵ_i so that τ_i is the time to cross this barrier. Thus $\tau_i = e^{\beta\epsilon_i}$, and the optimal path corresponds to the minimum $(\sum_i \tau_i)$ over all possible paths. Here $\beta = 1/kT$, where k is the Boltzmann constant and T is the temperature. When $\beta \to \infty$, only the largest τ_i dominates the sum. Thus $T \to 0$ (very low temperatures) corresponds to the strong disorder limit.

2.2 Dijkstra algorithm

This algorithm [9] is used in general to find the optimal path, when the weights are drawn from an arbitrary distribution. The search for the optimal path follows a procedure akin to "burning" where the "fire" starts from our chosen origin. At the beginning, all nodes are given a distance ∞ except the origin which is given a distance 0. At each step we choose the next unburned node which is nearest to the origin, and "burn" it, while updating the optimal distance to all its neighbors. The optimal distance of a neighbor is updated only if reaching it from the current burning node gives a total path length that is shorter than its current distance.

2.3 Ultrametric Optimization

Next we describe a numerical method for computing $\ell_{\rm opt}$ between any two nodes in strong disorder. We assign weights to all the links in the graph where the order of magnitude is taken from a uniform distribution [12]. This is accomplished by selecting $0 \leq \epsilon_i < 1$ from a uniform distribution, using a 48-bit random number generator, so that there are no two identical values of ϵ_i in a system of any size that we study. In this case $\Delta\epsilon \geq 2^{-48}$ and we can select $\beta \geq 2^{48}$ to guarantee the strong disorder limit. In the limit of strong disorder the sum of the weights is dominated by the largest value along the path. Next, we start from one node (the origin—see Fig. 1) and visit all the other nodes connected to the origin using a burning algorithm. If a node at distance ℓ_0 (from the origin) is being visited for the first time, this node will be assigned a list S_0 of weights τ_{0i}, $i = 1 \cdots \ell_0$ of the links by which we reach that node sorted in descending order,

$$S_0 = \{\tau_{01}, \tau_{02}, \tau_{03}, ...\tau_{0l_0}\}, \tag{1}$$

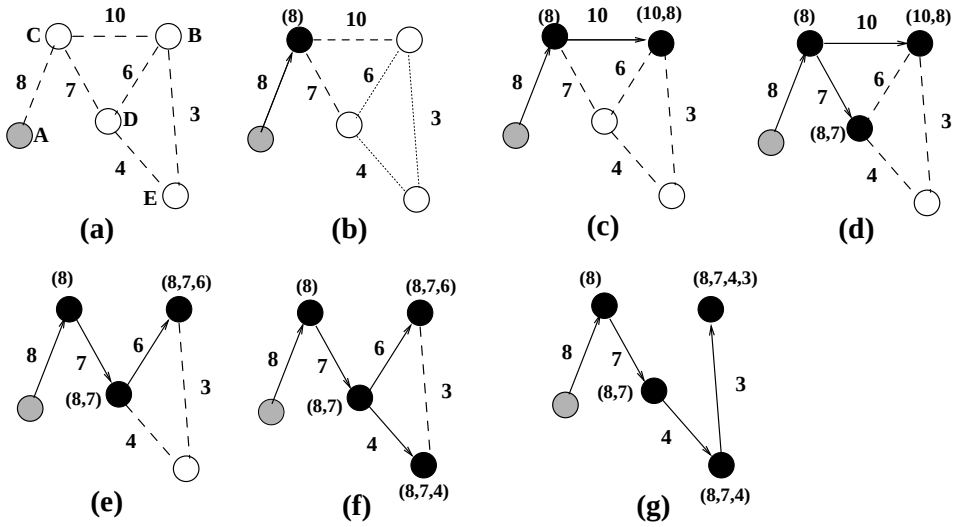

Fig. 1. In (a) we show schematically a network consisting of five nodes (A, B, C, D, and E). The links between them are shown in dashed lines. The origin (A) is marked in gray. All links were assigned random weights, shown beside the links. In (b) one node (C) has been visited for the first time (marked in black) and assigned the sequence (8) of length $\ell = 1$. The path is marked by a solid arrow. Notice that there is no other path going from the origin (A) to this node (C) so $\ell_{\rm opt} = 1$ for that path. In (c) another node (B) is visited for the first time (marked in black) and assigned the sequence $(10, 8)$ of length 2. The sequence has the information of all the weights of that path arranged in decreasing order. In (d) another node (D) is visited for the first time and assigned the sequence $(8, 7)$ of length 2. In (e), node (B) visited in (c) with sequence $(10, 8)$ is visited again with sequence $(8, 7, 6)$. The last sequence is smaller than the previous sequence $(10, 8)$ so that node (B) is reassigned the sequence $(8, 7, 6)$ of length 3. The new path is shown as a solid line. In (f) a new node (E) is assigned with sequence $(8, 7, 4)$. In (g) node (B) is reached for the third time and reassigned the sequence $(8, 7, 4, 3)$ of length 4. The optimal path for this configuration from A to B is denoted by the solid arrows in (g).

with $\tau_{0j} > \tau_{0j+1}$ for all j. If we reach a node for a second time by another path of length ℓ_1, we define for this path a new list S_1,

$$S_1 = \{\tau_{11}, \tau_{12}, \tau_{13}, \ldots \tau_{1l_1}\}, \tag{2}$$

and compare it with S_0 previously defined for this node.

Different sequences can have weights in common because some paths have links in common because of the loops, so it is not enough to identify the sequence by its maximum weight; in this case it must also be compared with the second maximum, the third maximum, etc. We define $S_p < S_q$ if there exists a value m, $1 \leq m \leq \min(\ell_p, \ell_q)$ such that

$$\tau_{pj} = \tau_{qj} \qquad \text{for} \qquad 1 \leq j < m \qquad \text{and}$$

$$\tau_{pj} < \tau_{qj} \qquad \text{for} \qquad j = m, \tag{3}$$

or if $\ell_q > \ell_p$ and $\tau_{pj} = \tau_{qj}$ for all $j \leq \ell_p$. If $S_1 < S_0$, we replace S_0 by S_1. The procedure continues until all paths have been explored and compared. At this point, $S_0 = S_{\rm opt}$, where $S_{\rm opt}$ is the sequence of weights for the optimal path of length $\ell_{\rm opt}$. A schematic representation of this ultrametric algorithm is presented in Fig. 1. This algorithm is slow and memory consuming since we have to keep track of a sequence of values and the rank. Using this method, we obtain systems of sizes up to 2^{12} nodes, typically 10^5 realizations of disorder.

2.4 Bombing Optimization

This algorithm allows to compute $\ell_{\rm opt}$ (and other relevant quantities) between any two nodes in strong disorder. This algorithm was first introduced by Cieplak et. al. [13] and is valid in the limit of strong disorder. Basically the algorithm does the following

(i) Sort the edges by descending weight.
(ii) If the removal of the highest weight edge will not disconnect A from B – remove it.
(iii) Go back to step ii until all edges have been processed.

Since the edge weights are random, so is the ordering. Therefore, in fact, one needs not even select edge weights and "bombing" algorithm can be replaced by simply removing randomly chosen edges one at a time, where an edge is not removed if its removal will breaks the connectivity between A and B. The final path left is the optimal path between A and B in the limit $\beta \to \infty$. This bombing algorithm is slow, since one must test the connectivity after removal of each link. To enhance the speed, we first find any path (selected path) on the network that connects two nodes. Then we remove links in random order. Only if the removed link belongs to the selected path, we check if the connectivity between the two nodes is still present. If the connectivity between the two nodes is destroyed, we restore the link because this link belongs to the optimal path and select another path that connects A and B. The best choice for the selected path is the minimal path, at a given concentration of links, because it is the shortest among all other existing paths between the two nodes, the probability to select a link from it is lower than the probability to select a link from any other path. The advantage of this procedure is that one has to test for connectivity only if the selected link belongs to the minimal path. Since checking the connectivity is the most time consuming part in the original "bombing' algorithm, with this improvement we could reach systems of sizes up to 2^{16} nodes and 10^5 realizations of weight disorder.

3 Strong Disorder

We begin with the analytical considerations in the case of strong disorder. This is believed to be relevant for computer and traffic networks, since the slowest link in communication networks determines the connection speed. To obtain the optimal path in the strong disorder limit, we present the following theoretical argument. It has been shown [13,12] that the optimal path for $\beta \to \infty$ between two nodes A and B on the network can be obtained by the bombing algorithm described in Section 2.4. This algorithm is based on randomly removing links. Since randomly removing links is a percolation process, the optimal path must be on the percolation backbone connecting A and B. Since the network is not embedded in space but has an infinite dimensionality, we expect from percolation theory that at criticality loops are not relevant. Thus, the shortest path at the percolation cluster at criticality must be the same as the optimal path.

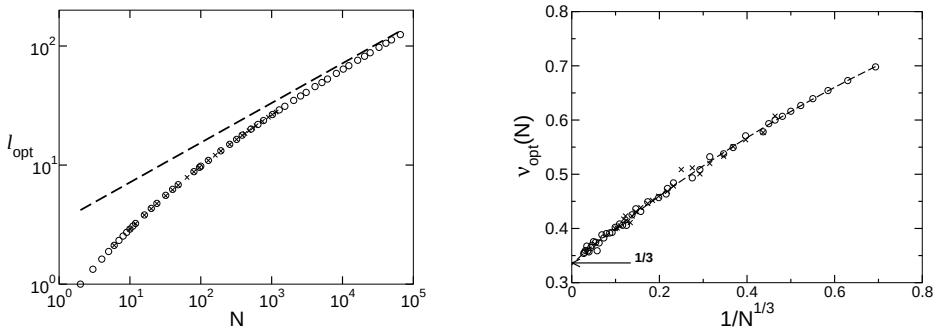

Fig. 2. Left - Plot of $\ell_{\rm opt}$ as a function of N in double logarithmic scale for the optimal path length in strong disorder using the two numerical methods discussed in the text: (i) results obtained using the "bombing" approach ($\circ$) and (ii) results obtained using the ultrametric approach ($\times$). The dashed line shows the slope 1/3. Right - Successive slopes $\nu_{\rm opt}(N)$ as a function of $1/N^{1/3}$ for the optimal path length in strong disorder using the two methods described in the text. The symbols denote the same as in Fig. 2. The dashed line is the quadratic fitting of the results showing that the extrapolated value of the effective exponent in the limit $N \to \infty$ approaches 1/3. This result coincides with our theoretical value $\nu_{opt} = 1/3$ asymptotically.

We begin by considering the case of the ER graph. At criticality, it is equivalent to percolation on the Cayley tree or percolation at the upper critical dimension $d_c = 6$. For the ER graph, it is known that the mass of the incipient infinite cluster S scales as $N^{2/3}$ [4]. This result can also be obtained in the framework of percolation theory for $d_c = 6$. Since $S \sim R^{d_f}$ and $N \sim R^d$ (where d_f is the fractal dimension and R the spatial diameter of the cluster), it follows that $S \sim N^{d_f/d}$ and for $d_c = 6$, $d_f = 4$ [14] we obtain $S \sim N^{2/3}$.

It is also known [14] that, at criticality, at the upper critical dimension, $S \sim$

$\ell_{\min}^{d_\ell}$ with $d_\ell = 2$, and thus

$$\ell_{\min} \sim \ell_{\rm opt} \sim S^{1/d_\ell} \sim N^{2/3d_\ell} \sim N^{\nu_{\rm opt}}, \tag{4}$$

where $\nu_{\rm opt} = 2/3d_\ell = 1/3$.

For SF networks, we can also use the percolation results at criticality. It was found [15] that $d_\ell = 2$ for $\lambda > 4$, $d_\ell = (\lambda-2)/(\lambda-3)$ for $3 < \lambda < 4$, $S \sim N^{2/3}$ for $\lambda > 4$, and $S \sim N^{(\lambda-2)/(\lambda-1)}$ for $3 < \lambda \leq 4$. Hence, we conclude that

$$\ell_{\min} \sim \ell_{\rm opt} \sim \begin{cases} N^{1/3} & \lambda > 4 \\ N^{(\lambda-3)/(\lambda-1)} & 3 < \lambda \leq 4 \end{cases}. \tag{5}$$

Thus $\nu_{opt} = 1/3$ for ER and SF with $\lambda > 4$, and $\nu_{opt} = (\lambda-3)/(\lambda-1)$ for SF with $3 < \lambda < 4$. Since for SF networks with $\lambda > 4$ the scaling behavior of $\ell_{\rm opt}$ is the same as for ER graphs and for $\lambda < 4$ the scaling is different, we can regard SF networks as a generalization of ER graphs.

Next we describe the details of our numerical simulations and show that the results agree with our theoretical predictions. We perform numerical simulations in the strong disorder limit by the method described in Section 2.4 for ER and SF networks. We also perform additional simulations for the case of strong disorder on ER networks using the ultrametric optimization algorithm (see Section 2.3) and find results identical to the results obtained by randomly removing links. In Fig. 2 (left) we show a double logarithmic plot of $\ell_{\rm opt}$ as a function of N for ER graphs. To evaluate the asymptotic value for $\nu_{\rm opt}$ we use for both approaches successive slopes, defined as centered differences of the values on Fig. 2. One can see from Fig. 2 (right) that their value approaches $1/3$ as $N >> 1$, supporting Eq. (4).

The theoretical considerations [Eqs. (4) and (5)] predict that SF graphs with $\lambda > 4$, are similar to ER with $\ell_{\rm opt} \sim N^{1/3}$, while for SF graphs with $3 < \lambda < 4$, $\ell_{\rm opt} \sim N^{(\lambda-3)/(\lambda-1)}$. Figure 3a shows data from numerical simulations supporting the linear behavior of $\ell_{\rm opt}$ versus $N^{1/3}$ for $\lambda \geq 4$. The quality of the linear fit becomes poor for $\lambda \to 4$. At this value, there are corrections due to logarithmic divergence of the second moment of the degree distribution, i.e., $\ell_{\rm opt} \sim N^{1/3}/\ln N$ (see Fig. 3b). Figure 3c shows results of simulations supporting the asymptotic linear behavior of $\ell_{\rm opt}$ versus $N^{(\lambda-3)/(\lambda-1)}$ for $3 < \lambda \leq 4$. Theoretically, as $\lambda \to 3$, $\nu_{\rm opt} = (\lambda-3)/(\lambda-1) \to 0$, and thus one can expect for $\lambda = 3$ a logarithmic dependence of $\ell_{\rm opt}$ versus N. Interestingly, for $2 < \lambda < 3$ our numerical results for the strong disorder limit suggest that $\ell_{\rm opt}$ scales faster than $\ln N$. The numerical results can be fit to $\ell_{\rm opt} \sim (\ln N)^{\lambda-1}$ (see Fig. 3d). Note that the correct asymptotic behavior may be different and this result represents only a crossover regime. The exact nature of the percolation cluster at $\lambda < 3$ is not clear yet, since in this regime the transition does not

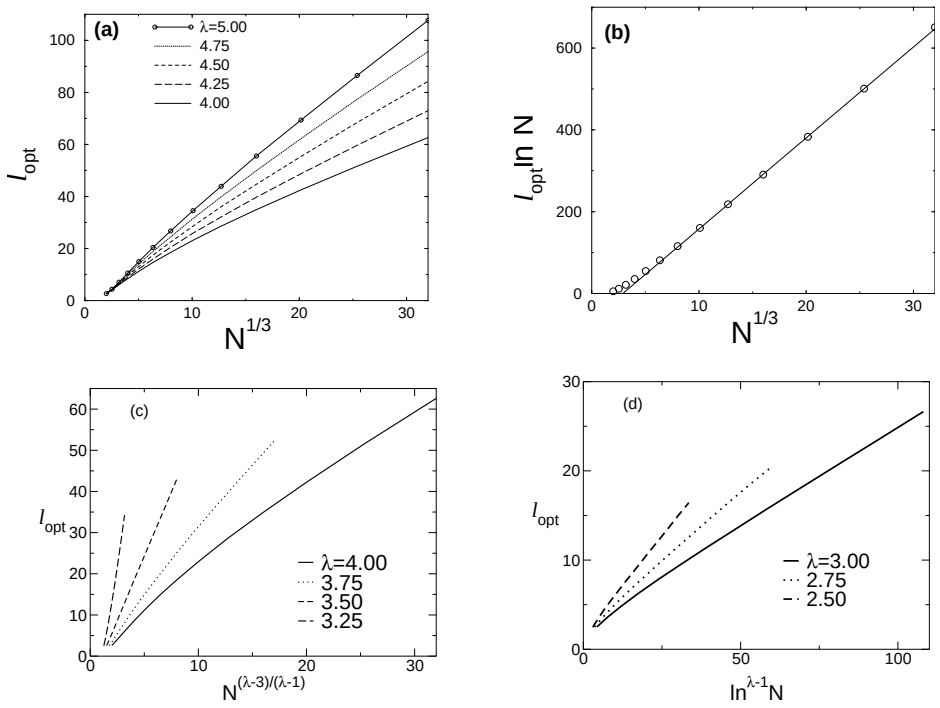

Fig. 3. Results of numerical simulations. (a) The dependence of $\ell_{\rm opt}$ on $N^{1/3}$ for $\lambda \geq 4$. (b) The dependence of $\ell_{\rm opt}/\ \ln N$ on $N^{1/3}$ for $\lambda = 4$. (c) The dependence of $\ell_{\rm opt}$ on $N^{(\lambda-3)/(\lambda-1)}$ for $3 < \lambda < 4$. (d) The dependence of $\ell_{\rm opt}$ on $\ln N$ for $\lambda \leq 3$.

occur at a finite concentration [16]. We obtain similar results for SF networks where the weights are associated with nodes instead of links.

4 Weak Disorder

When $\beta = 1/kT \to 0$, all the τ_i contributes to the total cost. Thus $T \to \infty$ (very high temperatures) corresponds to weak disorder limit. We expect that the optimal path length in the weak disorder case will not be considerably different from the shortest path, as found also for regular lattices [17] and random graphs [18]. Thus we expect that the scaling for the shortest path will also be valid for the optimal path in weak disorder, but with a different prefactor depending on the details of the graph and the type of disorder. We simulate weak disorder by selecting $0 \leq \tau_i < 1$ from a uniform distribution. To compute $\ell_{\rm opt}$ we use the Dijkstra algorithm (See Section 2.2)[9]. The scaling of the length of the optimal path in WD for ER, is shown in Fig .4(a). Here we plot $\ell_{\rm opt}$ as a function of $\ln N$ for $\langle k \rangle = 4$. The weak disorder does not change the scaling behavior of $\ell_{\rm opt}$ on ER compared to ℓ_{min}.

For SF networks, the behavior of the optimal path in the weak disorder limit is shown in Fig. 4(b) for different degree distribution exponents λ. Here we

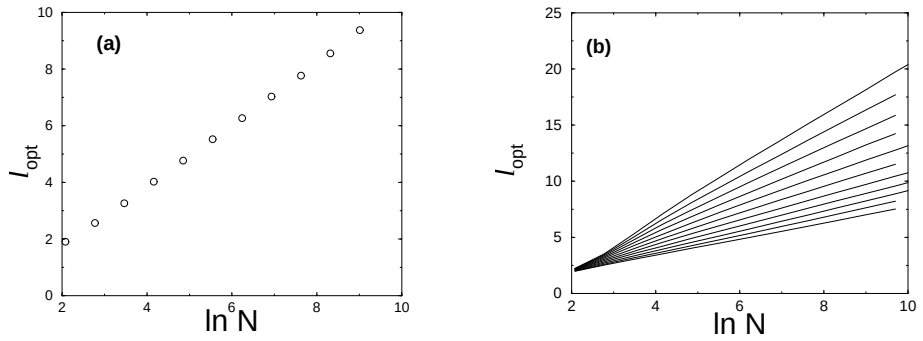

Fig. 4. Results of numerical simulations. (a) The linear dependence of $\ell_{\rm opt}$ on $\ln N$ for ER graphs in the weak disorder case for $< k >= 4$. (b) The dependence of $\ell_{\rm opt}$ on $\ln N$ for SF graphs in the weak disorder case for various values of λ. The different curves represent different values of λ from 2.5 (bottom) to 5 (top).

plot $\ell_{\rm opt}$ as a function of $\ln N$. All the curves seem to have linear asymptotes. This result is analogous to the behavior of the shortest path $\ell_{\rm min} \sim \ln N$ for $3 < \lambda < 4$ and ER. However, for $2 < \lambda < 3$, $\ell_{\rm min}$ scales as $\ln \ln N$ [7] while $\ell_{\rm opt}$ is significantly larger and scales as $\ln N$ (Fig. 3b). Thus weak disorder does not change the universality class of the length of the optimal path except in the case of "ultra-small" worlds $2 < \lambda < 3$, where $\ell_{\rm opt} \sim \exp(\ell_{min})$.

5 Transition from Weak to Strong Disorder

Consider the case for finite β (finite temperature). In this case we expect a transition on the length of the optimal path from strong disorder scaling characteristic to weak disorder depending on the system size as well as on the value of β. In order to study this transition we have to use an implementation of disorder that can be tuned to realize narrow distributions of link costs (WD) as well as broad distributions of link costs (SD). The procedure that we adopt to implement the disorder is as follows [13,19,12,20]. One assigns to each link i of the network a random number r_i, uniformly distributed between 0 and 1. The cost associated with link i is then $\tau_i \equiv \exp(\beta r_i)$, where β controls the broadness of the distribution of link costs. The parameter β represents the strength of disorder. The limit $\beta \to \infty$ is the strong disorder limit, where a single link dominates the cost of the path. We are going to call the length of the optimal path in the SD regime ℓ_∞, i.e, $l_{opt} \equiv l_\infty$.

For strong disorder, $\ell_\infty \sim N^{\nu_{\rm opt}}$ [12], where $\nu_{\rm opt} = 1/3$ for ER random networks and for SF networks with $\lambda > 4$, where λ is the exponent characterizing the power law decay of the degree distribution. For SF networks with $3 < \lambda < 4$, $\nu_{\rm opt} = (\lambda - 3)/(\lambda - 1)$ (see Section 3). For weakly disordered ER networks and for SF networks with $\lambda > 3$, $\ell_{\rm opt} \sim \ln N$ (see Section 4). Porto et al. [19]

considered the optimal path transition from weak to strong disorder for 2-D and 3-D lattices, and found a crossover in the scaling properties of the optimal path that depends on the disorder strength β, as well as the lattice size L. Similar to regular lattices, there exists for any finite β, a crossover network size $N^*(\beta)$ such that for $N \ll N^*(\beta)$, the scaling properties of the optimal path are in the strong disorder regime while for $N \gg N^*(\beta)$, the network is in the weak disorder regime. So the optimal length $\ell_{\rm opt}(\beta)$ depends on β as well as on N [21]. In the following we use instead of N, the optimal path in the ultrametric limit, ℓ_∞ which is related to N as $\ell_\infty \sim N^{\nu_{opt}}$. There exists a crossover length $\ell^*(\beta)$, corresponding to the crossover network size $N^*(\beta)$, such that the scaling properties of $\ell_{\rm opt}(\beta)$ are: (i) for $\ell_\infty \ll \ell^*(\beta)$ in SD, and (ii) for $\ell_\infty \gg \ell^*(\beta)$ in WD. The measure used to study the crossover is $\ell_{\rm opt}(\beta)/\ell_\infty$. It is found [21] that for ER networks, $N^*(\beta) \sim \beta^3$, while for SF networks with $3 \leq \lambda \leq 4$, $N^*(\beta) \sim \beta^{(\lambda-1)/(\lambda-3)}$.

6 Summary

In this paper, we review recent work on the scaling of the average optimal path length $\ell_{\rm opt}$ in a disordered network. There are two scaling regimes of l_{opt} corresponding to the regimes of weak and strong disorder. For ER networks and SF networks with $\lambda > 4$, $\ell_{\rm opt} \sim \ln N$ in the weak disorder regime while $\ell_{opt} \sim N^{1/3}$ in the strong disorder regime. For SF networks with $3 < \lambda < 4$, $\ell_{opt} \sim \ln N$ in the weak disorder regime while $\ell_{opt} \sim N^{\frac{\lambda-3}{\lambda-1}}$ in the strong disorder regime. For SF networks with $2 < \lambda < 3$, $l_{opt} \sim \ln N$ in the weak disorder regime while $l_{opt} \sim \ln^{\lambda-1} N$ in the strong disorder regime. The scaling behavior of ℓ_{opt} in the strong disorder regime for ER and SF networks with $\lambda > 3$ is obtained analytically using percolation theory [22]. Also mentioned briefly is work on the crossover in the scaling properties of $\ell_{\rm opt}$ from the weak disorder regime to the strong disorder regime. This work is discussed in detail in Sreenivasan et. al. (this volume).

Acknowledgments

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[21] For more details refer S. Sreenivasan et. al., this issue.

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