Infinitesimal analysis 88-503 Moed A 29 july '24

Each problem is worth 22 points. All answers must be fully justified by giving complete proofs. **The writing in the exam book must be fully legible and neat.**

1. This question deals with saturation (revaya).
   (a) Give a detailed formulation of the property of countable saturation.
   (b) Use countable saturation to deduce the existence of positive infinite members of ${}^*\mathbb{R}$.

2. This question deals with the LSEQ transformation.
   (a) Give a definition of the LSEQ transformation when applied to a formula ϕ.
   (b) Apply LSEQ to Ψ_1 and determine whether the new formula is true, where Ψ_1 is the formula
$$(\forall r \in \mathbb{R})(\exists x \in \mathbb{R}) \ \arctan r < x.$$

3. Apply the LSEQ transformation to Ψ_i in
   (a) the formula $(\forall \epsilon \in \mathbb{R}^+)\, \Psi_2(\epsilon)$ and determine whether the new formula is true, where $\Psi_2(\epsilon)$ is the formula
$$(\forall x \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall y \in \mathbb{R}^+) \ \big(|x-y| < \delta \ \to \ \sqrt{x} - \sqrt{y} < \epsilon\big).$$
   (b) the formula $(\forall \epsilon \in \mathbb{R}^+)\, \Psi_3(\epsilon)$ and determine whether the new formula is true, where $I = \left(0, \frac{\pi}{2}\right)$ and $\Psi_3(\epsilon)$ is the formula
$$(\forall x \in I)\, (\exists \delta \in \mathbb{R}^+)\, (\forall y \in I) \ \big(|x-y| < \delta \ \to \ \tan x - \tan y < \epsilon\big).$$

4. Let ${}^*\mathbb{Q}$ be the $*$-transform of $\mathbb{Q}$.
   (a) Use transfer so show that every hyperrational in ${}^*\mathbb{Q}$ is a ratio of hyperintegers.
   (b) Let L denote the set of finite hyperrationals, and let I denote the set of infinitesimal hyperrationals. Determine the quotient L/I.

5. This question deals with compactness.
   (a) Give a detailed formulation of the hyperreal characterisation of compactness of sets in $\mathbb{R}$.
   (b) Use the hyperreal characterisation of compactness to show that a nested decreasing sequence of nonempty compact sets in $\mathbb{R}$ has a common point.

Good Luck!