DIFFERENTIAL GEOMETRY 88-526–FINAL EXAM–MOED B
24 february '09
PLEASE JUSTIFY ALL ANSWERS
Duration of exam: 2.5 hours (150 minutes)

1. This problem deals with curves in Euclidean space.
   (a) Define the arclength parameter s of a curve;
   (b) Consider the curve $\alpha(t) = (4\cos t, 5 - 5\sin t, -3\cos t)$. Find an arclength parametrisation s of the curve;
   (c) Calculate the curvature $k(s)$ of the curve in part (b).

2. This problem deals with surfaces in Euclidean space. Throughout this problem, assume that $g_{ij} = L_{ij} = 0$ if $i \neq j$.
   (a) Define principal curvatures κ_1 and κ_2;
   (b) Choose a suitable basis and express the Weingarten map of M as a matrix;
   (c) Express the ratio κ_1/κ_2 as a functio fo the coefficients of the first and second fundamental forms;
   (d) Calculate the ratio κ_1/κ_2 for the surface of revolution obtained by rotating the parabola $x = z^2 + \frac{1}{4}$ around the z-axis.

3. In coordinates $(u^1, u^2) = (x, y)$, let $f(x, y) = \frac{2}{y}$, and consider the metric

$$f(x, y)^2(dx^2 + dy^2).$$

   (a) Calculate the symbol Γ^1_{11} of the metric;
   (b) Calculate the Gaussian curvature function $K = K(x, y)$ of the metric.

4. This problem deals with surfaces.
   (a) Derive an expression for Γ^k_{ij} in terms of the metric coefficients g_{ij};
   (b) Prove that the expression $\frac{\partial}{\partial u^k}\left(\Gamma^\ell_{ij}x_\ell + L_{ij}n\right)$ is symmetric in j and k; Explain the relation between L_{ij} and L^k_ℓ;
   (c) Express $L_{i[j}L^k_{\ell]}$ in terms of the metric coefficients.

5. Consider the torus T^2_0 in $\mathbb{R}^3$ parametrized by

$$x(\theta, \phi) = ((5 + \cos\phi)\cos\theta, (5 + \cos\phi)\sin\theta, \sin\phi).$$

   (a) Define the stable norm $\| \; \|$ in 1-dimensional homology $H_1(T^2; \mathbb{Z})$ of a torus T^2;
   (b) Calculate all the successive minima λ_1 and λ_2 of $H_1(T^2_0; \mathbb{Z})$ with respect to the norm $\| \; \|$;
   (c) Express the conformal parameter $\tau_0 = \tau(T^2_0)$ in terms of an integral;
   (d) Does every Riemannian torus T^2 conformally equivalent to T^2_0 necessarily satisfy the inequality

$$\mathrm{sys}_1(\mathbb{T})^2 \leq \mathrm{area}(\mathbb{T})?$$

   and provide an explanation.

GOOD LUCK!