88-826 Differential Geometry, moed B
Bar Ilan University, Prof. Katz
Date: 7 september '22

Each of 4 problems is worth 25 points; the bonus problem is 8 points

All answers must be justified by providing detailed definitions and complete explanations and proofs

1. Let M be a closed connected n-dimensional manifold.
   - (a) Consider a metric g on M. Give detailed definitions of the volume of a 2-cycle in M and of the stable norm.
   - (b) Give a detailed definition of the stable 2-systole of g.
   - (c) Give a detailed formulation of the duality between the stable norm and the comass norm.

2. Consider a closed connected 10-dimensional Riemannian manifold (M, g). Assume that $b_2(M) = 1$ and that a class $\omega \in L^2_{dR}(M)$ satisfies $\omega^{\cup 5} \neq 0$.
   - (a) Let $\eta \in \omega$ be a representative differential 2-form. Find a relation between $\eta_p^{\wedge 5}$ and $\|\eta\|_p^5$.
   - (b) Find a lower bound for $|\int_M \eta^{\wedge 5}|$ with proof.
   - (c) Estimate the integral $\int_M \eta^{\wedge 5}$ in terms of the comass of η as well as the total volume $\mathrm{vol}(M)$ of M.
   - (d) Prove an optimal upper bound for the ratio $\mathrm{stsys}_2(g)^5/\mathrm{vol}(g)$.

3. Let T^2 be a torus with a Riemannian metric g. Suppose T^2 contains an annulus $A = \mathbb{R}/\mathbb{Z} \times I$ such that the class of $\mathbb{R}/\mathbb{Z}$ is nontrivial in $H_1(T^2; \mathbb{Z})$.
   - (a) Give a detailed definition of the capacity of the annulus A.
   - (b) Suppose the capacity of the annulus $A \subseteq T^2$ is C. Prove an optimal inequality relating $\mathrm{sys}_1(g)$, C, and $\mathrm{area}(g)$.
   - (c) Use the result of (b) to prove an optimal systolic inequality for the torus.

4. Let $n \geq 1$, and let $M_n = \mathbb{CP}^2 \times S^n$. Let g be a metric on M_n of total volume 1. Determine (with proof) for which n is there a uniform upper bound for the stable 2-systole of M_n, valid for all such metrics g.

5 (bonus). Let α be the area form of S^2, expressed away from the poles as $\alpha(\theta, \phi) = \sin\phi \, d\theta \wedge d\phi$. Let β be the area form of $\mathbb{CP}^1$, expressed in an affine neighborhood as $\beta(x, y) = \frac{dx \wedge dy}{(1+x^2+y^2)^2}$. Consider the manifold $X = S^2 \times \mathbb{CP}^1$. Let $r, s \in \mathbb{R}$, and consider the form $\gamma = r\alpha + s\beta$ on X. Determine (with proof) for which values of r, s the form γ is exact.

Good Luck!