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בס"ד

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February 24, 2014 Differential geometry 88-826 Homework 1

1. Consider the curve $\alpha(t) = 2\cos t, 2\sin t$ in the (u, v)-plane. Consider the derivation X on the space $\mathbb{D}_p$ of smooth functions $f \in \mathbb{D}_p$ near the point $p = (\sqrt{2}, \sqrt{2})$ given by $X(f) = \frac{d}{dt}(f(\alpha(t))|_{t=\pi/4}$. Express X as a linear combination of the partial derivatives $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ at the point p.

2. The volume of an open region $D \subset \mathbb{R}^3$ is calculated with respect to cylindrical coordinates (r, θ, z) using the volume element

$$dV = r\, dr\, d\theta\, dz.$$

Namely, an integral is of the form $\int_D dV = \iiint r\, dr\, d\theta\, dz$.

(a) Find the volume of a right circular cone with height h and base a circle of radius b.

(b) evaluate the integral $\iiint\limits_E \sqrt{x^2 + y^2} z dV$ where E is the cylinder $x^2 + y^2 \le 1$, $0 \le z \le 2$.

(c) Find the volume of the object filling the region above the paraboloid $z = x^2 + y^2$ and below the plane $z = 1$.

3. Spherical coordinates (ρ, θ, ϕ) range between the bounds $0 \le \rho$, $0 \le \theta \le 2\pi$, and $0 \le \phi \le \pi$ (note the different upper bounds for θ and ϕ). The area of a spherical region D is calculated using a volume element of the form $dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$, so that the volume of a region D is $\int_D dV = \iiint\limits_D \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$.

(1) Find the volume of the region above the cone $\phi = \beta$ and inside the sphere of radius $\rho = c$.

(2) Find the integral $\iiint\limits_E x^2 + y^2 + z^2 dV$, where E is the sphere $x^2 + y^2 + z^2 = b^2$.

(3) Find the integral $\iiint\limits_E \frac{1}{x^2+y^2+z^2} dV$, where E is the region between two spheres: $a \le \rho \le b$.

4. Let $\delta^i{}_j$ be the Kronecker delta function on $\mathbb{R}^n$, where $i, j = 1, \ldots, n$, viewed as a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$. Evaluate the expression

$$\delta^i{}_j \delta^j{}_k \delta^k{}_i.$$

דיפרנציאלית ב-תרגיל 1

1

$$X(f) = \frac{d}{dt}\left(f(\underbrace{2\cos t}_{u}, \underbrace{2\sin t}_{v})\right) =$$

$$\frac{\partial f}{\partial u} \cdot \underbrace{\frac{\partial u}{\partial t}\Big|_{t=\pi/4}}_{-2\sin t|_{\pi/4} = -\sqrt{2}} + \frac{\partial f}{\partial v} \cdot \underbrace{\frac{\partial v}{\partial t}\Big|_{t=\pi/4}}_{2\cos t|_{\pi/4} = \sqrt{2}}$$

$$= -\sqrt{2}\,\frac{\partial f}{\partial u} + \sqrt{2}\,\frac{\partial f}{\partial v}$$

and therefore $\boxed{X = -\sqrt{2}\,\frac{\partial}{\partial u} + \sqrt{2}\,\frac{\partial}{\partial v}}$

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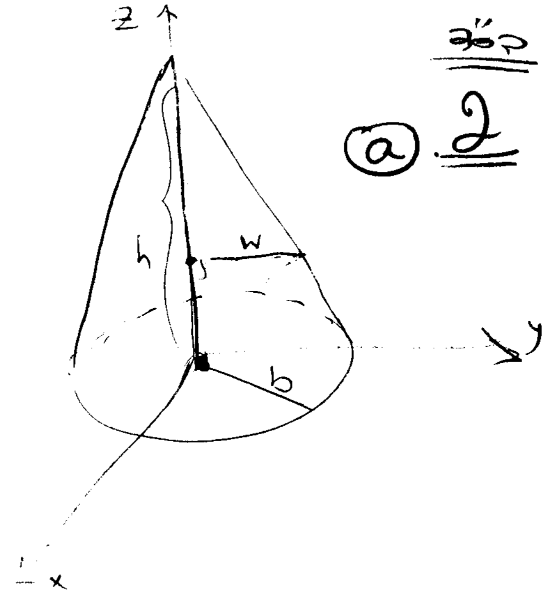
קטע

ⓐ **2**



$$\int_D dV = \int_0^h \int_0^{2\pi} \int_0^{w} r \, dr \, d\theta \, dz$$

תאלס

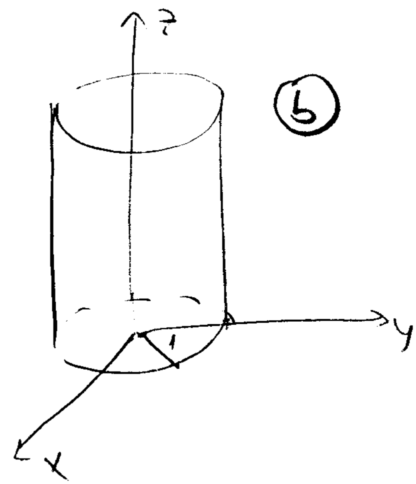
$$\frac{h}{h-z} = \frac{b}{w} \Rightarrow w = \frac{b(h-z)}{h}$$

$$= 2\pi \int_0^h \int_0^{\frac{b(h-z)}{h}} r \, dr \, dz$$

$$= 2\pi \int_0^h \left(\frac{r^2}{2} \Big|_0^{\frac{b(h-z)}{h}} \right) dz = 2\pi \int_0^h \frac{b^2 (h-z)^2}{2h^2} \, dz =$$

$$= \frac{\pi b^2}{h^2} \int_0^h (h-z)^2 \, dz = \frac{\pi b^2}{h^2} \cdot \frac{(h-z)^3}{-3} \Big|_0^h = \frac{\pi b^2}{h^2} \cdot \frac{h^3}{3} = \frac{\pi b^2 h}{3}$$

ⓑ



$$\iiint_E \underbrace{\sqrt{x^2+y^2}}_{r} \, z \, dV$$

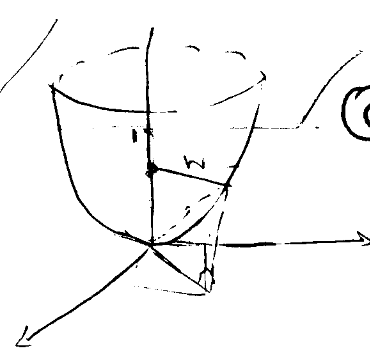
$$r = \sqrt{x^2+y^2}$$

$$= \iiint_E r z \, dV = \int_0^2 \int_0^{2\pi} \int_0^1 r z r \, dr \, d\theta \, dz$$

$$= 2\pi \int_0^2 z \left(\frac{r^3}{3} \Big|_0^1 \right) dz = \frac{2\pi}{3} \int_0^2 z \, dz$$

$$= \frac{2\pi}{3} \frac{z^2}{2} \Big|_0^2 = \frac{4\pi}{3}$$

ⓒ



$$vol = \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{z}} r \, dr \, d\theta \, dz$$

$$z = x^2+y^2$$
$$w^2 = x^2+y^2$$
$$\Downarrow$$
$$w = \sqrt{z}$$

$$= 2\pi \int_0^1 \left(\frac{r^2}{2} \Big|_0^{\sqrt{z}} \right) dz =$$

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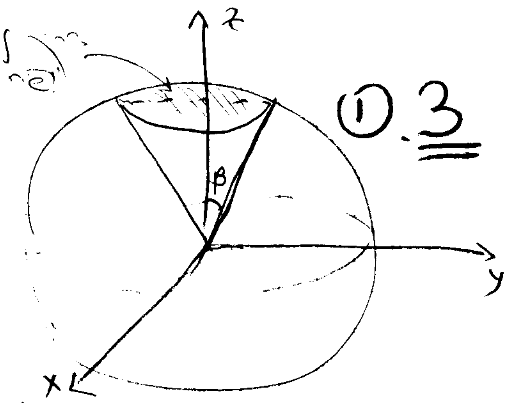
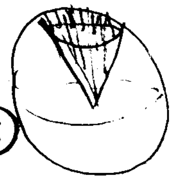
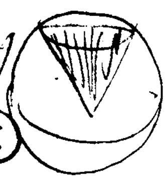
$$= 2\pi \int_0^1 \frac{z}{2}\, dz = 2\pi \cdot \frac{z^2}{4}\Big|_0^1 = \frac{\pi}{2}$$

①.3

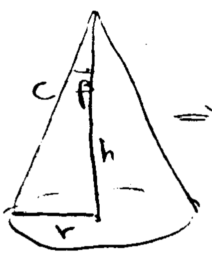


$\mathrm{vol}\, E = \mathrm{vol}$  I $- \mathrm{vol}$  II

I $\displaystyle\int_0^{\beta}\int_0^{2\pi}\int_0^{c} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi =$

$$= 2\pi \int_0^{\beta} \left(\frac{\rho^3}{3} \sin\phi \Big|_0^c \right) d\phi = \frac{2\pi c^3}{3} \left(-\cos\phi\right)\Big|_0^{\beta}$$

$$= \frac{2\pi c^3}{3} \left(-\cos\beta + 1\right)$$

II 

$\Rightarrow \sin\beta = \frac{r}{c} \Rightarrow r = c\sin\beta$

$\cos\beta = \frac{h}{c} \Rightarrow h = c\cdot\cos\beta$

לכן מ״ש שאלה 2 סעיף a נוכל

$$\text{II} = \frac{\pi c^3 \sin^2\beta \cdot \cos\beta}{3}$$

The intended region was above the surface of the cone. Hence the answer is simply I

$\mathrm{vol}\, E = \text{I} - \text{II}$ ובסה״כ

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$$\iiint_E x^2+y^2+z^2 \, dV \underset{x^2+y^2+z^2=\rho}{=} \int_0^{\pi}\int_0^{2\pi}\int_0^{b} \rho^2 \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= 2\pi \int_0^{\pi} \left(\frac{\rho^5}{5} \sin\phi \Big|_0^b \right) d\phi = \frac{2\pi b^5}{5} \int_0^{\pi} \sin\phi \, d\phi \underset{\text{כמו ב- ①}}{= 2}$$

$$= \frac{4\pi b^5}{5}$$

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$$\iiint_E \frac{1}{x^2+y^2+z^2} \, dV \underset{x^2+y^2+z^2=\rho^2}{=} \int_0^{\pi}\int_0^{2\pi}\int_a^{b} \frac{1}{\rho^2} \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= 2\pi \int_0^{\pi} (b-a) \sin\phi \, d\phi = 2(b-a)\pi \underbrace{\int_0^{\pi} \sin\phi \, d\phi}_{\text{כמו ב ① } = 2} = 4(b-a)\pi$$

(4)

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$$\delta^i_j \, \delta^j_k \, \delta^k_i = \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \delta^i_j \, \delta^j_k \, \delta^k_i$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \delta^j_k \, \delta^k_j$$

$$= \sum_{k=1}^{n} \delta^k_k = n$$

Good, but there is no need for the summation signs Σ. Simply use the formula $\delta^i{}_k \, \delta^k{}_j = \delta^i{}_j$, twice and then evaluate the trace.