WHO GAVE YOU THE CAUCHY-WEIERSTRASS TALE? THE DUAL HISTORY OF RIGOROUS CALCULUS

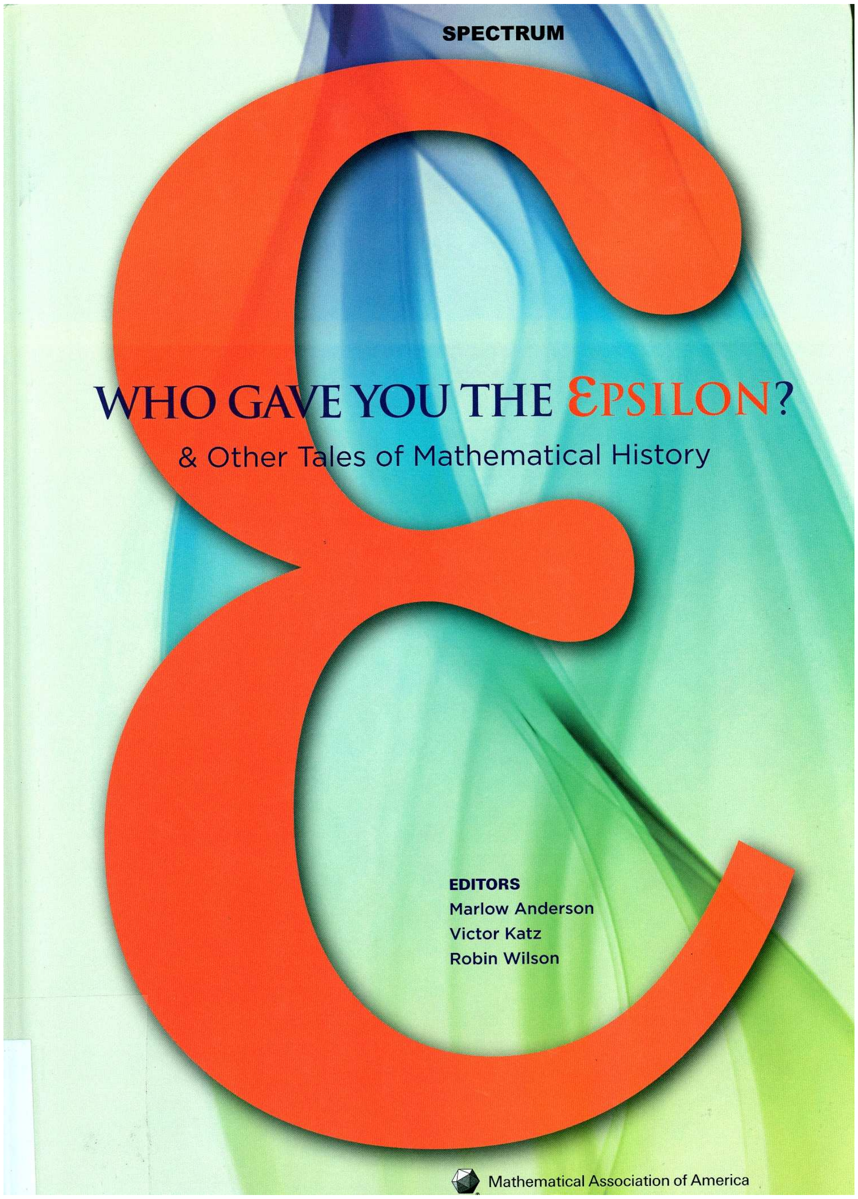


FIGURE 1. Epsilontics

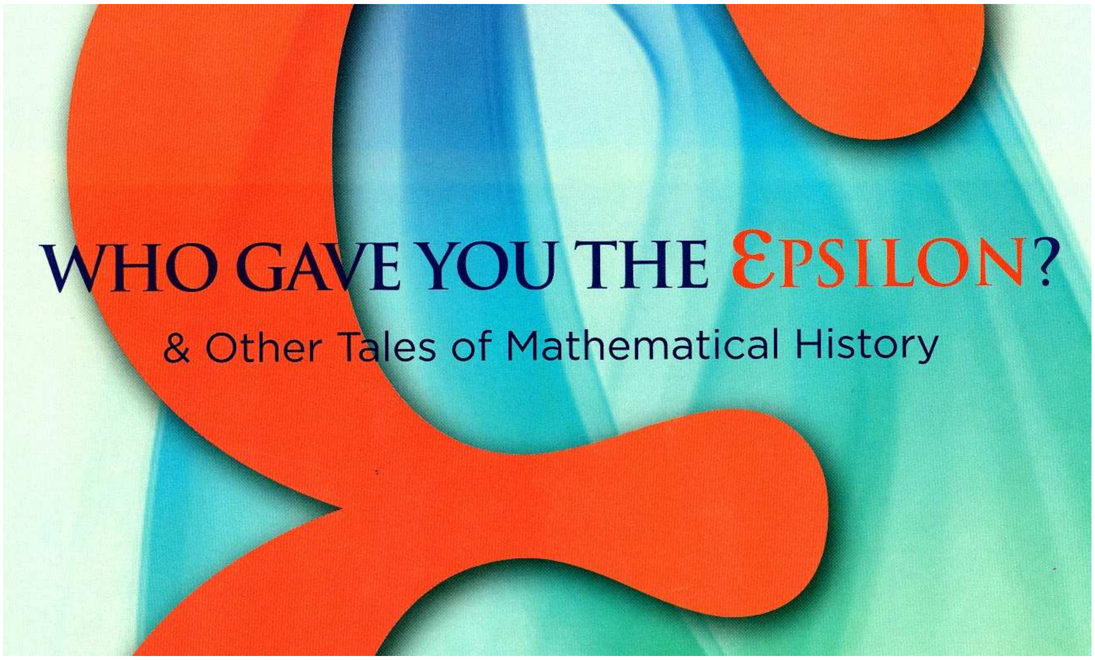


Figure 2. Epsilontics

1. Introduction

Joint work with:

- Karin Katz
- David Tall

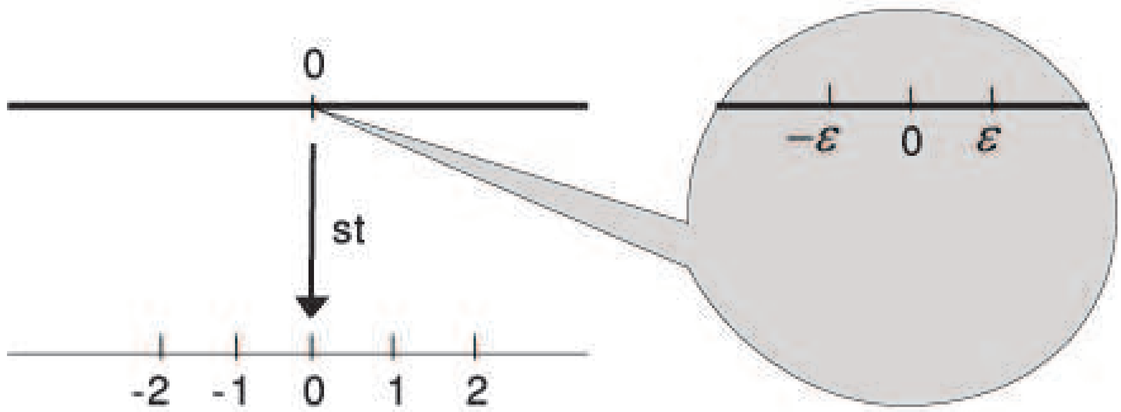


FIGURE 3. Zooming in on infinitesimal ϵ

Notation: Robinson's *standard part* "**st**".

2. Can one see an infinitesimal?

- A thin continuum
- A thick continuum

Johann Bernoulli's systematic use of infinitesimals.

- A-continuum (A for Archimedian property)
- B-continuum (B for Bernoulli)

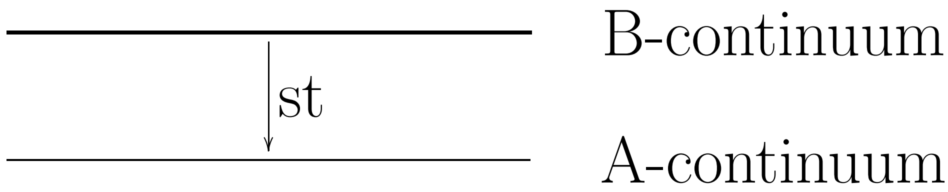


FIGURE 4. Taking standard part

Denote the B-continuum by the new symbol

$$\mathbb{I}\!\mathbb{R}$$

(pronounced "*thick-R*"), and its finite part,

$$\mathbb{I}\!\mathbb{R}_{<\infty}.$$

The map "st" sends each finite point x to the real point $\text{st}(x) \in \mathbb{R}$ infinitely close to x:

$$\begin{array}{c} \mathbb{I}\!\mathbb{R}_{<\infty} \\ \downarrow \text{st} \\ \mathbb{R} \end{array}$$

Here "st" collapses the *cluster* of x, back to x.

Robinson's answer to Berkeley's logical criticism:

Definition of derivative of $y = f(x)$ as

$$f'(x) = \mathrm{st}\left(\frac{\Delta y}{\Delta x}\right)$$

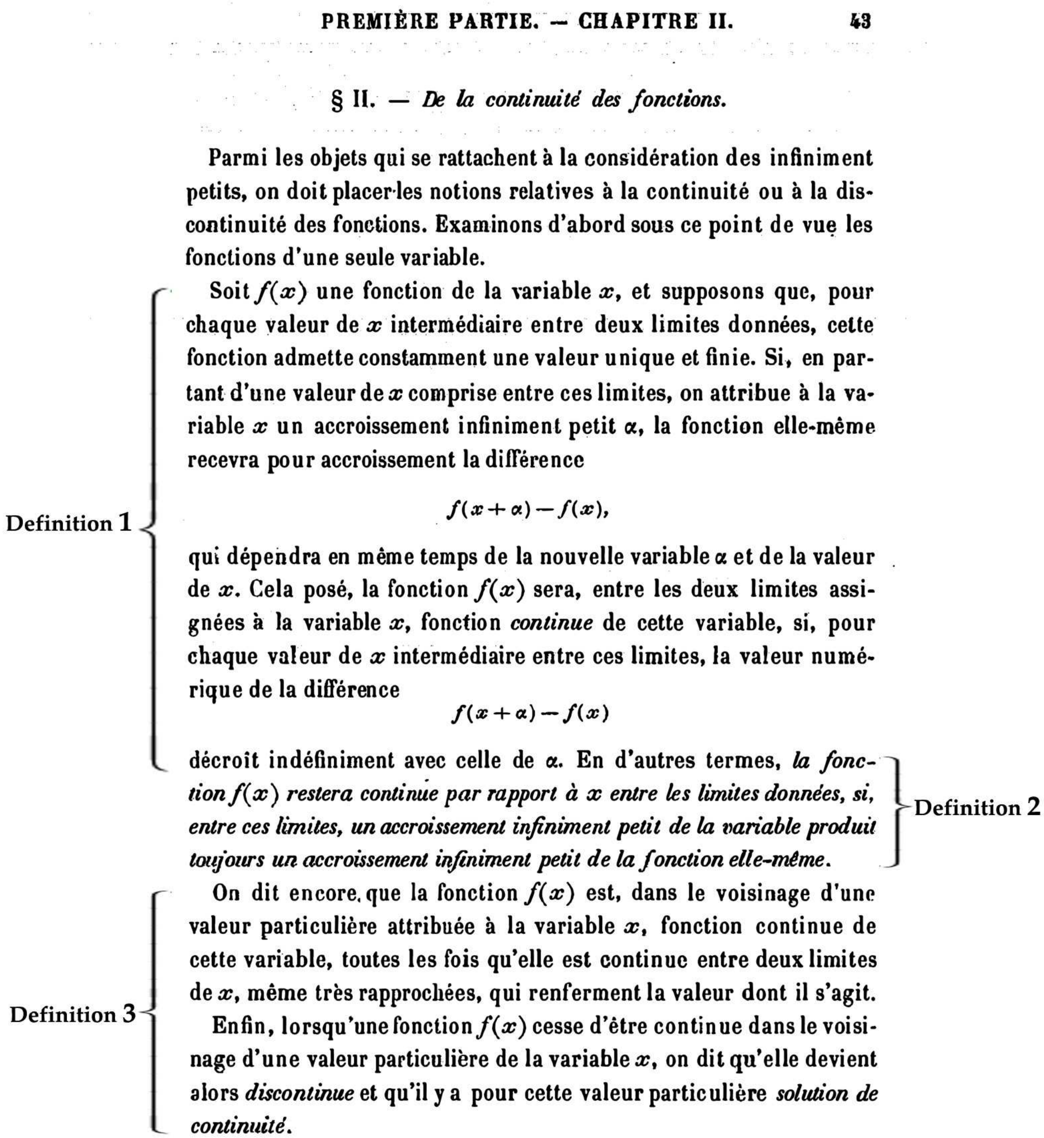
instead of $\frac{\Delta y}{\Delta x}$.

3. Cauchy's *Cours d'Analyse*

Cauchy's *Cours d'Analyse* dates from 1821.

- Cauchy gave 3 definitions of continuity (p. 43 of *Collected Works*).
- The last one is based on the first 2.
- Cauchy did **not** give any ϵ, δ definition of continuity.

How did Cauchy come to be thought of as a precursor of a Weierstrassian definition of continuity?



PREMIÈRE PARTIE. — CHAPITRE II. 43

§ II. — *De la continuité des fonctions.*

Parmi les objets qui se rattachent à la considération des infiniment petits, on doit placer les notions relatives à la continuité ou à la discontinuité des fonctions. Examinons d'abord sous ce point de vue les fonctions d'une seule variable.

Soit $f(x)$ une fonction de la variable x, et supposons que, pour chaque valeur de x intermédiaire entre deux limites données, cette fonction admette constamment une valeur unique et finie. Si, en partant d'une valeur de x comprise entre ces limites, on attribue à la variable x un accroissement infiniment petit α, la fonction elle-même recevra pour accroissement la différence

$$f(x+\alpha)-f(x),$$

qui dépendra en même temps de la nouvelle variable α et de la valeur de x. Cela posé, la fonction $f(x)$ sera, entre les deux limites assignées à la variable x, fonction *continue* de cette variable, si, pour chaque valeur de x intermédiaire entre ces limites, la valeur numérique de la différence

$$f(x+\alpha)-f(x)$$

décroît indéfiniment avec celle de α. En d'autres termes, *la fonction $f(x)$ restera continue par rapport à x entre les limites données, si, entre ces limites, un accroissement infiniment petit de la variable produit toujours un accroissement infiniment petit de la fonction elle-même.*

On dit encore que la fonction $f(x)$ est, dans le voisinage d'une valeur particulière attribuée à la variable x, fonction continue de cette variable, toutes les fois qu'elle est continue entre deux limites de x, même très rapprochées, qui renferment la valeur dont il s'agit.

Enfin, lorsqu'une fonction $f(x)$ cesse d'être continue dans le voisinage d'une valeur particulière de la variable x, on dit qu'elle devient alors *discontinue* et qu'il y a pour cette valeur particulière *solution de continuité.*

FIGURE 5. Cours d'Analyse

The structure of Cauchy's first two definitions is summarized in Table 1.

	independent variable increment (Δx)	dependent variable increment (Δy)
Cauchy's first definition	infinitesimal	variable tending to zero
Cauchy's second definition	infinitesimal	infinitesimal

TABLE 1. Cauchy's first two definitions of continuity in 1821

4. Berkeley's dual criticisms

Philosopher David Sherry (1987) analyzed George Berkeley's criticism as expressed in *The Analyst* (1734).

Sherry's dichotomy (details in paper upon request):

(1) the *metaphysical* criticism,
(2) the *logical* criticism.

Reformulation of Sherry's dichotomy in mathematical terms:

(1) (metaphysical) how can Archimedean continuum contain non-Archimedean elements (i.e., infinitesimals)?
(2) (logical) Berkeley's "*ghosts of departed quantities*" criticism of the definition of the derivative as an infinitesimal quotient.

Elaboration of logical criticism:

shift in hypothesis

How can a quantity (dx) at the same time retain a "*ghost*" ($dx \neq 0$), and be "*departed*" ($dx = 0$)?

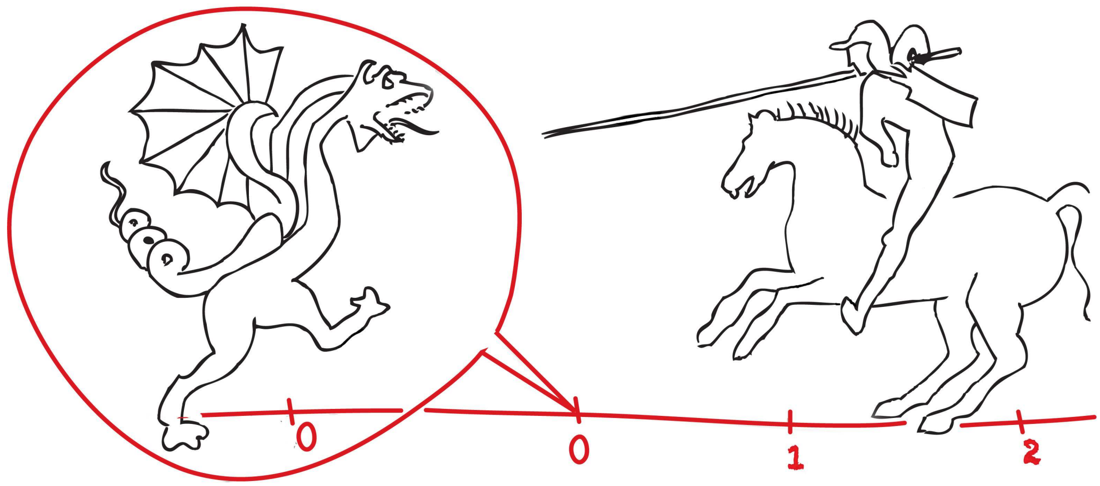


FIGURE 6. Reverend George's attempted slaying of the infinitesimal, following E. T. Bell and P. Uccello.

Response to Berkeley's *metaphysical criticism*: a presentation of a stratified (hierarchical) number system.

A-continuum is englobed inside a B-continuum:

$$\mathbb{R} \subset \mathrm{I\!I\!R},$$

a distinction already available in the 19th century (Carnot and Cauchy, see below).

Meanwhile, an adequate response to Berkeley's *logical criticism* was given in the 1960s by Robinson, as mentioned above.

5. Is an A-continuum unique?

Two distinct implementations of an A-continuum:

- the Stevin numbers $\mathbb{R}$, in the context of classical logic (incorporating the law of excluded middle);
- Brouwer's continuum built from "free-choice sequences", in the context of intuitionistic logic.

Here Simon Stevin (1548-1620) first constructed the usual numbers in 1585, in terms of arbitary decimals.

6. Is a B-continuum unique?

Philosopher John Lane Bell describes a dichotomy within the B-continuum.

Historically, there were two approaches to an B-continuum:

- by Leibniz,
- by B. Nieuwentijdt.

Nieuwentijdt favored nilpotent (nil-square) infinitesimals ϵ., In other words,

$$\epsilon^2 = 0.$$

J. Bell notes that the two intuitions are realized respectively

- (Leibniz) in Robinson's *hyperreals*;
- (Nieuwentijdt) in Lawvere's *smooth infinitesimal analysis*.

The latter theory relies on intuitionistic logic (Lawvere's infinitesimals are incompatible with classical logic).

7. Carnot and Cauchy: infinitesimal as a null sequence

Back to A-continuum *versus* (thick) B-continuum: In the world of Carnot and Cauchy, two types of quantities:

- "constant" quantities, versus
- "variable" quantities.

Here an infinitesimal being viewed as *generated* by a **null sequence**. See Swedish scholar Kajsa Bråting (2007).

8. A fruitful application

As noted by Freudenthal (1971) and Laugwitz (1989), Cauchy (1827) uses

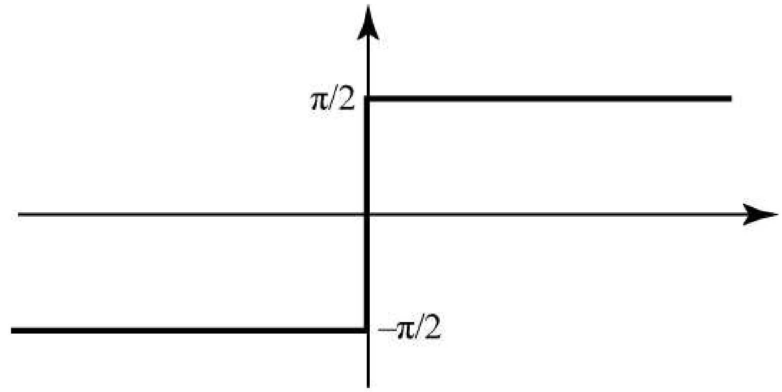


FIGURE 7. Heavyside function: primitive of Cauchy's Dirac delta function in 1827

infinitesimals α and ϵ to write down a Dirac delta function.

Cauchy applies it in Fourier analysis and evaluation of singular integrals. Cauchy wrote:

> *"Moreover one finds, denoting by α, ϵ two infinitely small numbers,"*

$$\frac{1}{2}\int_{a-\epsilon}^{a+\epsilon} F(\mu)\frac{\alpha\, d\mu}{\alpha^2+(\mu-a)^2} = \frac{\pi}{2}F(a)$$

Here α is, in modern terms, the "scale parameter" of the "Cauchy distribution".

9. Felix Klein on rivalry

Felix Klein (1908) commented on rivalry of such continua. Klein

(1) outlines the developments in real analysis associated with Weierstrass;
(2) notes that *"The scientific mathematics of today is built upon [such] developments"*; but
(3) *"an essentially different conception of infinitesimal calculus has been running parallel with this [conception] through the centuries;"*

(4) The different conception *"harks back to old metaphysical speculations concerning the* **structure of the continuum** *according to which this was made up of ...* **infinitely small** *parts."*

10. E. T. Bell's scalps

E. T. Bell (1945) waxed poetic about infinitesimals having been:

- *slain* [p. 246],
- *scalped* [p. 247], and
- *disposed of* [p. 290].

by the Bishop of Cloyne. Such *scalps of departed quantities* continue to litter the closets of otherwise serious studies.

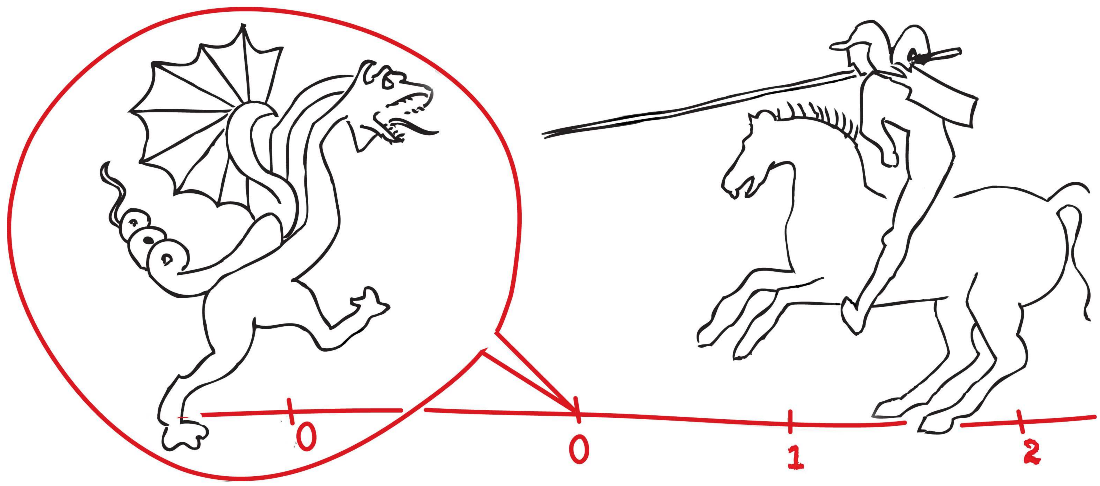


FIGURE 8. Reverend George's attempted slaying of the infinitesimal, following E. T. Bell and P. Uccello.

11. Cajori's horror

Historian Florian Cajori is a well-known Berkeley authority. Cajori (1917) describes infinitesimals as

- *fictitious* [p. 151];
- *horrible visions* [p. 152];
- he waxes eloquent about *"wonderful strides in the banishment of infinitely small quantities"* [p. 152]

- and about *"full and complete disavowal of infinitesimals"* [p. 153];
- he describes as a "point of merit" [of 18th century British conceptions], the *"abandonment of the use of infinitely small quantities"*;
- *"Not all English authors of the eighteenth century broke away from infinitesimals, but those who did were among the leaders"* [p. 153];
- *It was not until ... Weierstrass that infinitesimals were cast aside* [p. 154].

This latter claim is inaccurate: work on non-Archimedean number systems continued uninterrupted even after Weierstrass, see Ehrlich (2006).

12. Cauchy and Moigno

Cleric Moigno, a student of Cauchy's.

Historian Gert Schubring (2005) reports on a purportedly successful effort by Moigno "to pick apart" [p. 445] infinitesimal methodologies. Here Moigno puzzles over how an infinitesimal magnitude can possibly be less than its own half:

> these magnitudes, [assumed to be] smaller than any given magnitude, still have substance and are divisible[; however,] their existence is a chimera, since, necessarily greater than their half, their quarter, etc., they are not actually less than any given magnitude.

Moigno is focusing on Berkeley's metaphysical criticism, *not* the logical criticism.

As far as the metaphysical criticism is concerned, the solution was in Cauchy's

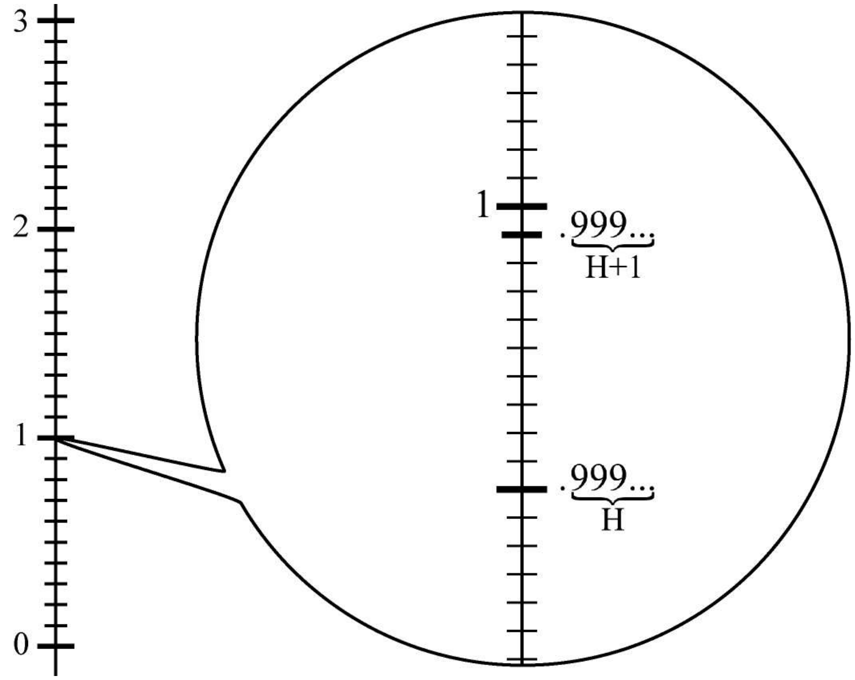


Figure 9. Three infinitely close hyperreals under a microscope

work: variable *versus* constant magnitude. Namely,

$$\text{A-continuum} \subset \text{B-continuum}.$$

13. To visualize an infinitesimal

An infinite tail of 9s with precisely a hyperfinite number H of 9s is shown

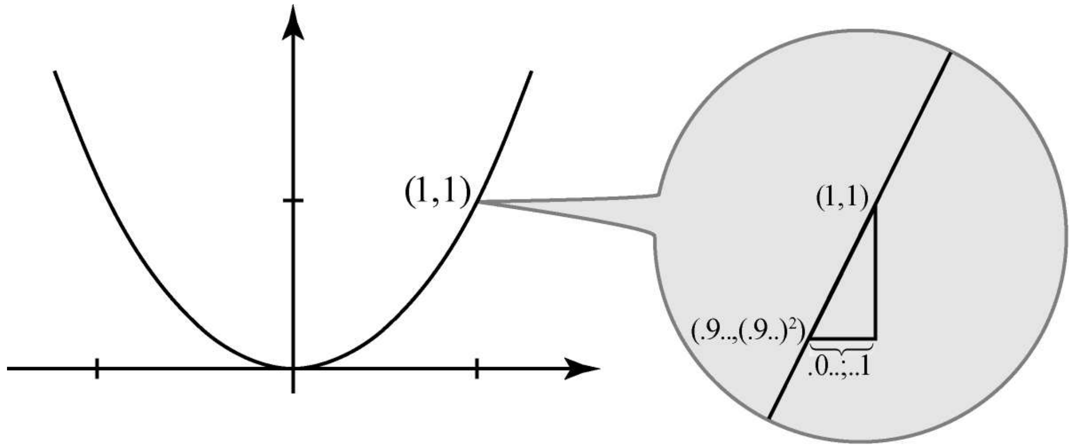


FIGURE 10. Enlargement of an infinitesimal segment of a parabola as calculation of the slope $f'(1)$ for $f(x) = x^2$.

in the figure, and similarly the number with $H + 1$ such 9s.

14. To apply an infinitesimal

An application to the calculation of the slope $f'(1)$ for $f(x) = x^2$.