Cauchy, Cours d'Analyse, 1821

2.2 On the continuity of functions.

[43] Among the objects related to the study of infinitely small quantities, we ought to include ideas about the continuity and the discontinuity of functions. In view of this, let us first consider functions of a single variable.

Let $f(x)$ be a function of the variable x, and suppose that for each value of x between two given limits, the function always takes a unique finite value. If, beginning with a value of x contained between these limits, we add to the variable x an infinitely small increment α, the function itself is incremented by the difference[6]

$$f(x+\alpha)-f(x),$$

which depends both on the new variable α and on the value of x. Given this, the function $f(x)$ is a *continuous* function of x between the assigned limits if, for each value of x between these limits, the numerical value of the difference

$$f(x+\alpha)-f(x)$$

Cauchy's definition of continuity: not an epsilon (or delta) in sight

decreases indefinitely with the numerical value of α. In other words, *the function $f(x)$ is continuous with respect to x between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself.*[7]

We also say that the function $f(x)$ is a continuous function of the variable x in a neighborhood of a particular value of the variable x whenever it is continuous between two limits of x that enclose that particular value, even if they are very close together.

Finally, whenever the function $f(x)$ ceases to be continuous in the neighborhood of a particular value of x, we say that it becomes discontinuous, and that there is *solution of continuity*[8] for this particular value.

[44] Having said this, it is easy to recognize the limits between which a given function of a variable x is continuous with respect to that variable. So, for example, the function $\sin x$, which takes a unique finite value for each particular value of the variable x, is continuous between any two limits of this variable, given that the numerical value of $\sin\left(\frac{1}{2}\alpha\right)$, and consequently that of the difference[9]

$$\sin(x+\alpha)-\sin x = 2\sin\left(\tfrac{1}{2}\alpha\right)\cos\left(x+\tfrac{1}{2}\alpha\right),$$

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[6] Cauchy defines continuity only on the interior of a bounded interval, and for the whole interval, not just at a single point. See [Grabiner 2005, p. 87] for more on this point. This passage is also cited in [DSB Cauchy, p. 136].

[7] [Grattan-Guinness 1970b] has suggested that Cauchy "stole" this and other ideas from Bolzano's paper of 1817. See also [Freudenthal 1971b, Jahnke 2003, p. 161, Grabiner 2005, pp. 9–12].

[8] This word "solution" takes an old meaning here; it means that continuity dissolves or disappears.

[9] To verify this formula, let $u = x+\frac{1}{2}\alpha$ and $v=\frac{1}{2}\alpha$, then apply the usual formula for $\sin(a+b)$ to the expression $\sin(u+v)-\sin(u-v)$.