Differential geometry

Mikhail G. Katz*

M. Katz, Department of Mathematics, Bar Ilan University, Ramat Gan 52900 Israel

Email address: katzmik@math.biu.ac.il

ABSTRACT. Lecture notes for the course 88-826 in differential geometry. Differentiable manifolds via coordinate charts and transition functions, differential k-forms, exterior differential complex, de Rham cohomology, Wirtinger inequality, systoles, Gromov's systolic inequality for complex projective spaces, generalisations. The course site is at `https://u.cs.biu.ac.il/~katzmik/88-826.html`

Contents

CHAPTER 1

Differentiable manifolds

(1) Course site `http://u.math.biu.ac.il/~katzmik/88-826.html`
(2) The final exam is 90% of the grade and the targilim 10%.
(3) The first homework assignment is due on 29 march '23.

1.1. Definition of differentiable manifold

An n-dimensional manifold is a set M possessing certain additional properties. A formal definition appears below as Definition 1.1.2. Here M is assumed to be covered by a collection of subsets (called coordinate charts or neighborhoods), typically denoted A or B, and having the following properties. For each coordinate neighborhood $A \subseteq M$ we have an injective map $u\colon A \to \mathbb{R}^n$ whose image

$$u(A) \subseteq \mathbb{R}^n$$

is an open set in $\mathbb{R}^n$. A coordinate chart is the pair

$$(A, u).$$

The maps are required to satisfy the following compatibility condition (see Definition 1.1.2). Let

$$u\colon A \to \mathbb{R}^n, \quad u = (u^i)_{i=1,\ldots,n}, \tag{1.1.1}$$

and similarly

$$v\colon B \to \mathbb{R}^n, \quad v = (v^\alpha)_{\alpha=1,\ldots,n} \tag{1.1.2}$$

be a pair of coordinate charts. Whenever the overlap $A \cap B$ is nonempty, it has a nonempty image $v(A \cap B)$ in Euclidean space. Both $u(A)$ and $u(A \cap B)$, etc., are assumed to be open subsets of $\mathbb{R}^n$.

Definition 1.1.1. Let v^{-1} be the inverse map of the map v of (1.1.2).

Thus v^{-1} is a map from the image (in $\mathbb{R}^n$) of the injective map v back to M. Restricting to the subset $v(A \cap B) \subseteq \mathbb{R}^n$, we obtain a one-to-one map between Euclidean domains:

$$\phi = u \circ v^{-1}\colon v(A \cap B) \to \mathbb{R}^n \tag{1.1.3}$$

from an open set $v(A \cap B) \subseteq \mathbb{R}^n$ to $\mathbb{R}^n$.

Similarly, the map $v \circ u^{-1}$ from the open set $u(A \cap B) \subseteq \mathbb{R}^n$ to $\mathbb{R}^n$ is one-to-one. We can now state the formal definition.

DEFINITION 1.1.2 (Smooth manifold). A *smooth n-dimensional manifold* M is a union

$$M = \cup_{\alpha \in I} A_\alpha,$$

where I is an index set, together with injective maps $u_\alpha \colon A_\alpha \to \mathbb{R}^n$, satisfying the following compatibility condition: the map ϕ of (1.1.3) is differentiable for all choices of coordinate neighborhoods $A = A_\alpha$ and $B = A_\beta$ (where $\alpha, \beta \in I$) as above.

DEFINITION 1.1.3. [Transition map] The map $\phi = u \circ v^{-1}$ is called a *transition map* or transition function.[1] The collection of coordinate charts as above is called an *atlas* for the manifold M.

DEFINITION 1.1.4. A 2-dimensional manifold is called a *surface*.

1.2. Topology on M

We define a topology on M as follows.

DEFINITION 1.2.1. The coordinate charts induce a topology on M by imposing the usual conditions:

(1) If $S \subseteq \mathbb{R}^n$ is an open set then the inverse image $v^{-1}(S) \subseteq M$ is defined to be open;[2]
(2) arbitrary unions of open sets in M are open;
(3) finite intersections of open sets are open.

REMARK 1.2.2. We will usually assume that M is connected. Given the manifold structure as above, connectedness of M is equivalent to path-connectedness[3] of M.

Identifying the overlap[4] $A \cap B \subseteq M$ with a subset of $\mathbb{R}^n$ by means of the coordinates (u^i), we can think of the map v as given by n real-valued functions

$$v^\alpha(u^1, \ldots, u^n), \quad \alpha = 1, \ldots, n. \tag{1.2.1}$$

These functions will be used to define the classes of smoothness of the manifold M in Section 1.5.

[1] funktsiat maavar or haatakat maavar

[2] Here $v^{-1}(S)$ is by definition the set of points $x \in M$ such that $v(x) \in S$. If S happens to be disjoint from the image of v then by definition $v^{-1}(S) = \varnothing$.

[3] kshir-mesila

[4] chafifa

1.3. Dependent and independent variables

We present an alternative formulation of the condition on transition functions. The formulation is in terms of dependent and independent variables and slightly more detailed notation. We retain the notation $u = (u^1, \ldots, u^n)$ and $v = (v^1, \ldots, v^n)$ for the coordinate variables in the respective $\mathbb{R}^n$, and denote by

$$\mathcal{U}\colon A \to \mathbb{R}^n \quad \text{and} \quad \mathcal{V}\colon B \to \mathbb{R}^n$$

the two charts. We let $C = A \cap B$. The transition map $\phi = \mathcal{V} \circ \mathcal{U}^{-1}$ is defined in the set $\mathcal{U}(C) \subseteq \mathbb{R}^n$, and its inverse ϕ^{-1} is defined in $\mathcal{V}(C)$. If we view $u = (u^1, \ldots, u^n)$ as the independent variables and v as the dependent variables, then the dependence $v^\alpha(u^1, \ldots, u^n)$ for each $\alpha = 1, \ldots, n$ is expressed by the components of the transition map ϕ. In particular, we can write

$$\frac{\partial \phi}{\partial u^i} = \left(\frac{\partial v^1}{\partial u^i}, \ldots, \frac{\partial v^n}{\partial u^i} \right)$$

for each i. If v is viewed as the independent variable and u as the dependent variable then the dependence $u = u(v)$ is given by the transition function $\psi = \phi^{-1}$. In particular, one can write

$$\frac{\partial \psi}{\partial v^\alpha} = \left(\frac{\partial u^1}{\partial v^\alpha}, \ldots, \frac{\partial u^n}{\partial u^\alpha} \right)$$

for each index α.

1.4. Metrizability

There are some non-Hausdorff examples that are pathological from the viewpoint of differential geometry, such as the following.

Example 1.4.1. Let X be two copies of $\mathbb{R}$ glued along an open halfline of $\mathbb{R}$. Then X satisfies the compatibility condition of Definition 1.1.2.

To rule out such examples, the simplest condition is that of *metrizability* of M: the topology generated by open balls of the metric (distance function) on M coincides with the topology underlying the differentiable structure as in Definition 1.2.1. See e.g., Example 1.7.1 and Theorem 1.8.6.[5] For relation to relativity see [**Sachs**]. Examples of manifolds M will be given in Sections 1.6, 1.7, and 1.8.

[5] For connected manifolds, metrizability implies separability; see Gauld [**3**].

1.5. Hierarchy of smoothness of manifold M

The manifold condition stated in Definition 1.1.2 can be reformulated as the requirement that the n real-valued functions $v^\alpha(u^1, \ldots, u^n)$ appearing in (1.2.1) are all smooth. What is the precise meaning of smoothness?

Definition 1.5.1. The usual hierarchy of smoothness (of functions), denoted C^k (or C^∞, or C^{an}), in Euclidean space generalizes to manifolds as follows. Here the conditions listed are assumed to be satisfied for all coordinate charts.

(1) For $k = 1$ a manifold M is C^1 if and only if all n^2 partial derivatives
$$\frac{\partial v^\alpha}{\partial u^i}, \quad \alpha = 1, \ldots, n,\ i = 1, \ldots, n$$
exist and are continuous.

(2) The manifold M is C^2 if all n^3 second partial derivatives
$$\frac{\partial^2 v^\alpha}{\partial u^i \partial u^j}$$
exist and are continuous.

(3) The manifold M is C^k if all the n^{k+1} partial derivatives
$$\frac{\partial^k v^\alpha}{\partial u^{i_1} \cdots \partial u^{i_k}}$$
exist and are continuous.

(4) The manifold M is C^∞ if for each $k \in \mathbb{N}$, all partial derivatives
$$\frac{\partial^k v^\alpha}{\partial u^{i_1} \cdots \partial u^{i_k}}$$
exist.

(5) The manifold M is C^{an} if for each $k \in \mathbb{N}$, all the functions
$$v = v(u^1, \ldots, v^n)$$
are real analytic functions.

The last condition is of course the strongest one of the five listed.

1.6. Open submanifolds, Cartesian products

The notion of open and closed set in M is inherited from Euclidean space via the coordinate charts as in Definition 1.2.1.

Definition 1.6.1. An open subset $C \subseteq M$ of a manifold M is itself a manifold, called an open submanifold of M. The differentiable structure on C is obtained by the restriction of the coordinate map $u = (u^i)$ of (A, u) for all charts (A, u) in M. The restriction will be denoted

$$u\restriction_{A\cap C}.$$

Example 1.6.2. Let $\mathrm{Mat}_{n,n}(\mathbb{R})$ be the set of square matrices with real coefficients. This is linearly identified with Euclidean space of dimension n^2, and is therefore a manifold.

Theorem 1.6.3. *Define a subset* $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathrm{Mat}_{n,n}(\mathbb{R})$ *by setting*

$$\mathrm{GL}(n, \mathbb{R}) = \{X \in \mathrm{Mat}_{n,n}(\mathbb{R})\colon\ \det(X) \neq 0\}.$$

Then $\mathrm{GL}(n, \mathbb{R})$ *is an open submanifold.*

Proof. The determinant function is a polynomial in the entries x_{ij} of the matrix X. Therefore it is a continuous function of the entries, which are the coordinates in $\mathrm{Mat}_{n,n}$. Thus $\mathrm{GL}(n, \mathbb{R})$ is the inverse image of the open set $\mathbb{R} \setminus \{0\}$ under a continuous map, and is therefore an open set, hence a manifold with respect to the restricted atlas. $\square$

Definition 1.6.4. The *determinantal variety* is the complement D of $\mathrm{GL}(n, \mathbb{R})$ in $\mathrm{Mat}_{n,n}(\mathbb{R})$. It is the closed set consisting of matrices of zero determinant:

$$\begin{aligned} D &= \mathrm{Mat}_{n,n}(\mathbb{R}) \setminus \mathrm{GL}(n, \mathbb{R}) \\ &= \{A\colon\ \det(A) = 0\}. \end{aligned}$$

Remark 1.6.5. The set D for $n \geq 2$ is not a manifold.

The proof of the following theorem is straightforward.

Theorem 1.6.6. *Let M and N be two differentiable manifolds of dimensions m and n. Then the Cartesian product $M \times N$ is a differentiable manifold of dimension $m + n$. The differentiable structure is defined by coordinate neighborhoods of the form $(A \times B, u \times v)$, where (A, u) is a coordinate chart on M, while (B, v) is a coordinate chart on N. Here the function $u \times v$ on $A \times B$ is defined by*

$$(u \times v)(x, y) = (u(x), v(y))$$

for all $x \in A, y \in B$.

1.7. Circle, tori

We give some additional examples of manifolds.

Theorem 1.7.1. *The circle $S^1 = \{(x, y) \in \mathbb{R}^2\colon\ x^2 + y^2 = 1\}$ is a differentiable manifold.*

PROOF. We will provide an explicit atlas for the circle, in four steps.

Step 1. Let $A^+ \subseteq S^1$ be the open upper halfcircle

$$A^+ = \{a = (x, y) \in S^1 \colon y > 0\}.$$

Consider the coordinate chart (A^+, u), namely

$$u \colon A^+ \to \mathbb{R}, \tag{1.7.1}$$

defined by setting $u(x, y) = x$ (vertical projection to the x-axis).

Step 2. Consider also the open lower halfcircle

$$A^- = \{a = (x, y) \in S^1 \colon y < 0\}.$$

It provides a coordinate chart (A^-, u) where the coordinate u is defined by the same formula (1.7.1).

Step 3. We define the right halfcircle

$$B^+ = \{a = (x, y) \in S^1 \colon x > 0\},$$

yielding a coordinate chart (B^+, v) where

$$v(x, y) = y, \tag{1.7.2}$$

and similarly for B^-.

Step 4. The theorem results from the Proposition 1.7.2 below on the transition functions ϕ. □

PROPOSITION 1.7.2. *The four charts A^+, A^-, B^+, and B^- cover the circle S^1 and define a differentiable structure on S^1.*

PROOF. It is clear that the charts cover the circle. Let us determine the transition functions.

Step 1. The transition function between A^+ and B^+ is calculated as follows. Note that in the overlap $A^+ \cap B^+$ one has both $x > 0$ and $y > 0$. Let us calculate the transition function $u \circ v^{-1}$. It follows from (1.7.2) that the map v^{-1} sends the point $y \in \mathbb{R}^1$ to the point

$$(\sqrt{1 - y^2}, y) \in S^1,$$

and then the coordinate map u sends the point $(\sqrt{1 - y^2}, y)$ to its first coordinate $\sqrt{1 - y^2} \in \mathbb{R}^1$.

Step 2. The composed map $\phi = u \circ v^{-1}$ given by

$$\phi(y) = \sqrt{1 - y^2} \tag{1.7.3}$$

is the transition function in this case.[6] The function (1.7.3) is smooth and analytic for all $y \in (0,1)$. Similar remarks apply to the remaining transition functions. It follows that the circle is an analytic manifold of dimension 1, modulo checking the Hausdorff condition.

Step 3. Finally we discuss the metrizability condition (see Section 1.4). We define a distance function on S^1 by setting

$$d(p,q) = \arccos\langle p, q\rangle \tag{1.7.4}$$

This gives a metric on S^1 having all the required properties. It follows that S^1 is metrizable. □

Example 1.7.3 (Tori). We apply Theorem 1.6.6.

(1) The torus $\mathbb{T}^2 = S^1 \times S^1$ is a 2-dimensional manifold.
(2) the n-torus $\mathbb{T}^n = S^1 \times \cdots \times S^1$ (product of n copies of the circle) is an n-dimensional manifold, for all $n \geq 1$.
(3) The circle itself is thought of as the 1-dimensional torus $\mathbb{T}^1$.

Example 1.7.4 (Spheres). The unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ admits an atlas similar to the case of the circle. The distance function is defined by the same formula (1.7.4) as for the circle, establishing metrizability in the general case.

1.8. Projective spaces

Another example of a manifold that has fundamental importance in both differential and algebraic geometry is the projective space, defined as follows. In this section we will mainly consider the real case. For the complex case, see Corollary 1.8.7 and further links there. Complex projective spaces are dealt with in more detail in Section 7.2.

To define the real projective space, we first let

$$X = \mathbb{R}^{n+1} \setminus \{0\}$$

be the set of $(n+1)$-tuples $x = (x^0, \ldots, x^n)$ distinct from the origin.

[6] In more detail, let $C = A^+ \cap B^+$ be the quarter circle in the first quadrant. Denote by $p\colon C \to \mathbb{R}$ the projection to the x-axis, and by $q\colon C \to \mathbb{R}$ the projection to the y-axis. Let $\phi = p \circ q^{-1}$ be the transition function. If y is viewed as the independent variable then $x = \phi(y) = \sqrt{1-y^2}$ and $\frac{dx}{dy} = \phi'(y) = \frac{-2y}{\sqrt{1-y^2}}$. When x is viewed as the independent variable then similar formulas exist with ϕ^{-1} in place of ϕ. A similar picture emerges in the higher-dimensional case, where the derivatives $\frac{dx}{dy}$ and $\frac{dy}{dx}$ are replaced by partial derivatives $\frac{\partial v^\alpha}{\partial u^i}$, as in Section 1.5.

Definition 1.8.1. An equivalence relation $\sim$ between elements $x, y \in X$ is defined by setting $x \sim y$ if and only if there is a real number $t \neq 0$ such that $y = tx$, i.e.,

$$y^i = tx^i, \quad i = 0, \ldots, n. \tag{1.8.1}$$

Definition 1.8.2. Denote by $[x]$ the equivalence class of $x \in X$.

Definition 1.8.3 (Homogeneous coordinates). $(x^0, \ldots, x^n)$ are the *homogeneous coordinates* of the point $[x]$.

Lemma 1.8.4. *For every $x \in X$, we have $[x] = [-x]$.*

Proof. This is immediate from the choice $t = -1$ of the scalar. $\square$

Definition 1.8.5. The real projective space, denoted $\mathbb{RP}^n$, is the collection of equivalence classes $[x]$, i.e.,

$$\mathbb{RP}^n = \{[x] \colon x \in X\}.$$

Theorem 1.8.6. *The space $\mathbb{RP}^n$ admits a natural structure of a smooth n-dimensional manifold.*

Proof. To show that the set $\mathbb{RP}^n$ is a manifold, we need to exhibit an atlas.

Step 1. We define coordinate neighborhoods A_k, where $k = 0, \ldots, n$ by setting

$$A_k = \{[x] \colon x^k \neq 0\}. \tag{1.8.2}$$

We will now define the coordinate pair (A_k, u_k), where $u_k \colon A_k \to \mathbb{R}^n$, namely the corresponding coordinate chart. We set

$$u_k(x) = \left(\frac{x^0}{x^k}, \ldots, \frac{x^{k-1}}{x^k}, \frac{x^{k+1}}{x^k}, \ldots, \frac{x^n}{x^k}\right). \tag{1.8.3}$$

Here formula (1.8.3) is valid since division by x^k is allowed in the neighborhood A_k by condition (1.8.2). The coordinate chart u_k is well-defined because if $x \sim y$ as in formula (1.8.1), then $u_k(x) = u_k(y)$ by canceling out the scalar t in the numerator and denominator.

Step 2. Let us calculate the transition maps between charts A_k for $k = i$ and $k = j$. We let $u = u_i$ and $v = u_j$. Assume for simplicity that $i < j$. We wish to calculate the transition map $\phi = u \circ v^{-1}$ associated with the overlap[7] $A_i \cap A_j$. Take a point

$$z = (z^0, \ldots, z^{n-1}) \in \mathbb{R}^n$$

[7]chafifa

in the image of v. Since we are working with the condition $x^j \neq 0$, we can rescale the homogeneous coordinates $(x^0, \ldots, x^n)$ (see Definition 1.8.3) so that $x^j = 1$. Thus we can represent $v^{-1}(z)$ by the $(n+1)$-tuple

$$v^{-1}(z) = (z^0, \ldots, z^{j-1}, 1, z^j, \ldots, z^{n-1}). \tag{1.8.4}$$

Now we apply $u = u_i$ to (1.8.4). The transition map $\phi(z) = u \circ v^{-1}(z)$ has the form

$$\phi(z) = \left(\frac{z^0}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots \frac{z^{j-1}}{z^i}, \frac{1}{z^i}, \frac{z^j}{z^i}, \ldots, \frac{z^{n-1}}{z^i} \right) \tag{1.8.5}$$

since $i < j$.

Step 3. All functions appearing as components of the transition map (1.8.5) are rational functions. Therefore the transition maps are smooth and analytic. Thus $\mathbb{RP}^n$ is an analytic differentiable manifold.

Step 4. Let us check the metrizability condition (see Section 1.4). For unit vectors $p, q \in X$ we set

$$d(p, q) = \arccos |\langle p, q \rangle|. \tag{1.8.6}$$

Note that due to the use of the absolute value in (1.8.6), we have $d(-p, q) = d(p, q)$, which is consistent with the fact that $[p] = [-p]$ (see Lemma 1.8.4). For arbitrary $p, q \in X$ we use the formula

$$d(p, q) = \arccos \frac{|\langle p, q \rangle|}{|p|\,|q|}. \tag{1.8.7}$$

Formula (1.8.7) provides a distance function on $\mathbb{RP}^n$ with all the required properties, showing that $\mathbb{RP}^n$ is metrizable. □

Note that p and $-p$ represent the same point in projective space.

COROLLARY 1.8.7. *The complex projective space* $\mathbb{CP}^n$ *is a smooth* $2n$-*dimensional manifold.*

PROOF. We use $\mathbb{C}$ in place of $\mathbb{R}$ to define the coordinate neighborhoods A_k as in formula (1.8.2), so that each A_k is a copy of $\mathbb{C}^n$. The transition functions defined by (1.8.5) are rational functions which are therefore smooth in their domain of definition. An additional piece of information for $\mathbb{CP}^n$ is the complex structure i.e., the endomorphism J dealt with in Theorem 7.2.3. To address the metrizability issue, we note that the space carries the natural Fubini–Study metric (8.1.1) dealt with in more detail in Section 8.1. □

1.9. Derivations, Leibniz rule

Let M be a differentiable manifold as defined in Sections 1.1 and 1.4. The tangent space, denoted T_pM, at a point $p \in M$ is intuitively the collection of all tangent vectors at the point p.[8] In modern differential geometry, a tangent vector can be defined via derivations; see Definition 1.9.4.

To define the tangent space, we first define the function ring $\mathbb{D}_p$ as follows.

Definition 1.9.1. Let $p \in M$. Let

$$\mathbb{D}_p = \{f \colon f \in C^\infty\}$$

be the ring of C^∞ real-valued functions f defined in an (arbitrarily small) open neighborhood of $p \in M$.

Definition 1.9.2. The ring operations in $\mathbb{D}_p$ are pointwise multiplication fg and pointwise addition $f+g$, where we choose the intersection of the domains of f and g as the domain of the new function (respectively sum or product). Thus, we set

$$(fg)(x) = f(x)\, g(x)$$

for all points x at which both functions are defined, and similarly for $f+g$.

Choose local coordinates $(u^1, \ldots, u^n)$ near the point $p \in M$. Then a function f defined near p can be thought of as a function of n variables, $f(u^1, \ldots, u^n)$. The following is proved in multivariate calculus.

Theorem 1.9.3. *A partial derivative $\frac{\partial}{\partial u^i}$ at the point p is a linear form, or 1-form, denoted $\frac{\partial}{\partial u^i} \colon \mathbb{D}_p \to \mathbb{R}$ on the space $\mathbb{D}_p$, satisfying the Leibniz rule*

$$\left.\frac{\partial (fg)}{\partial u^i}\right|_p = \left.\frac{\partial f}{\partial u^i}\right|_p g(p) + f(p) \left.\frac{\partial g}{\partial u^i}\right|_p \tag{1.9.1}$$

for all $f, g \in \mathbb{D}_p$.

Formula (1.9.1) can be written briefly as

$$\tfrac{\partial}{\partial u^i}(fg) = \tfrac{\partial}{\partial u^i}(f)\, g + f \tfrac{\partial}{\partial u^i}(g)$$

keeping in mind that both sides are evaluated only at the point p (*not* in a neighborhood of the point). Formula (1.9.1) motivates the following more general definition of a derivation at $p \in M$.

[8] A preliminary notion of a tangent space, or plane, to a surface is developed in introductory courses based on a Euclidean embedding of the surface; see e.g., `http://u.math.biu.ac.il/~katzmik/egreglong.pdf` (course notes for the course 88-201).

DEFINITION 1.9.4. A *derivation* X at the point $p \in M$ is a linear form

$$X\colon\ \mathbb{D}_p \to \mathbb{R}$$

on the space $\mathbb{D}_p$ satisfying the Leibniz rule:

$$X(fg) = X(f)g(p) + f(p)X(g) \tag{1.9.2}$$

for all $f, g \in \mathbb{D}_p$.

Note that the definition is coordinate-free. Chevalley gave such a definition in 1946 apparently for the first time.[9]

DEFINITION 1.9.5. The tangent space to M at p is the space of all derivations at p.

We will clarify the structure of the tangent space in Section 2.1.

[9] See `https://mathoverflow.net/a/491046`

CHAPTER 2

Derivations, tangent and cotangent bundles

2.1. The space of derivations

In Section 1.1 we defined the notion of a smooth manifold M. In Section 1.9 we gave a coordinate-free definition of a derivation X at a point $p \in M$. Here X is a linear form on the space of smooth functions $\mathbb{D}_p$ such that X satisfies the Leibniz rule at p. It turns out that the space of derivations is spanned by partial derivatives. Namely, we have the following theorem.

THEOREM 2.1.1. *Let M be an n-dimensional manifold. Let $p \in M$. Then the collection of all derivations at p is a vector space of dimension n.*

PROOF IN CASE $n = 1$. The proof is essentially an application of Taylor's formula. We will prove the result in the case $n = 1$. For example, one could think of the 1-dimensional manifold $M = \mathbb{R}$ with the standard smooth structure. Thus we have a single coordinate u in a neighborhood of a point $p \in M$ which can taken to be 0, i.e., $p = 0$. We argue in four steps as follows.

Step 1. Let $X \colon \mathbb{D}_p \to \mathbb{R}$ be a derivation at p. To prove the theorem, it suffices to show that X is proportional to the derivative $\frac{d}{du}$. Consider the constant function $1 \in \mathbb{D}_p$. Let us determine $X(1)$. We have $X(1) = X(1 \cdot 1) = 2X(1)$ by the Leibniz rule. Therefore $X(1) = 0$. Similarly for any constant a we have $X(a) = aX(1) = 0$ by linearity of X.

Step 2. Now consider the monic polynomial $u = u^1$ of degree 1, viewed as a linear function $u \in \mathbb{D}_{p=0}$. We evaluate the derivation X at the element $u \in \mathbb{D}_p$ and set $c = X(u)$. Thus $c \in \mathbb{R}$.

Step 3. By the Taylor formula, every function $f \in \mathbb{D}_{p=0}$ can be written as $f(u) = a + bu + g(u)u, \ \ a, b \in \mathbb{R}$, where g is smooth and $g(0) = 0$.

Recall that $f'(0) = b$. Therefore we have by linearity and Leibniz rule

$$\begin{aligned} X(f) &= X(a + bu + g(u)u) \\ &= bX(u) + X(g)u(0) + g(0) \cdot c \\ &= bc + 0 + 0 = c\,\frac{d}{du}(f). \end{aligned}$$

Step 4. The formula $(\forall f \in \mathbb{D}_p) \quad X(f) = c\frac{d}{du}(f)$ established in Step 3, means that derivation X coincides with the derivation $c\frac{d}{du}$. Hence the space of derivations is 1-dimensional and spanned by the derivation $\frac{d}{du}$, proving the theorem in the case $n = 1$.

The case of general n is treated similarly using a Taylor formula with partial derivatives. $\square$

We recall Definition 1.9.5.

DEFINITION 2.1.2. The *tangent space* T_pM to M at p is is the space of derivations at p.

2.2. Tangent bundle of a smooth manifold

Let M be a differentiable manifold. In Section 2.1 we defined the tangent space T_pM at $p \in M$ as the space of derivations at p. Next, we define the tangent bundle of M.

DEFINITION 2.2.1. The underlying set of the *tangent bundle*, denoted TM, of an n-dimensional manifold M is the disjoint union of all tangent spaces T_pM as p ranges through M, or in formulas:

$$TM = \bigsqcup_{p \in M} T_pM.$$

THEOREM 2.2.2. *The tangent bundle TM of a smooth n-dimensional manifold M has a natural structure of a smooth manifold of dimension $2n$.*

The proof appears below. First we recall some notational conventions.

DEFINITION 2.2.3 (Einstein summation convention). The rule is that whenever a product contains a symbol with a lower index and another symbol with the *same* upper index, take summation over this repeated index.

REMARK 2.2.4. The index i in $\frac{\partial}{\partial u^i}$ is considered to be a lower index since it occurs in the denominator.

We will use such notation in the proof of the theorem and elsewhere in this text.

PROOF OF THEOREM 2.2.3. We coordinatize TM locally using $2n$ coordinate functions as follows. By Theorem 2.1.1, a tangent vector v at a point p decomposes as

$$v = v^i \frac{\partial}{\partial u^i}$$

(with respect to the Einstein summation convention). In a coordinate neighborhood $A \subseteq M$, we have coordinates $(u^1, \ldots, u^n)$. We combine the $(u^1, \ldots, u^n)$ with the *components* v^i of tangent vectors $v \in T_pM$, with respect to the basis $(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \ldots, \frac{\partial}{\partial u^n})$, namely $v = v^i \frac{\partial}{\partial u^i}$. We obtain a string of $2n$ coordinates

$$(u^1, \ldots, u^n, v^1, \ldots, v^n). \tag{2.2.1}$$

Coordinates (2.2.1) of the pair (p, v) parametrize a neighborhood of the tangent bundle TM. It can be checked that the transition functions are smooth, showing that TM is a $(2n)$-dimensional manifold. □

DEFINITION 2.2.5 (Diffeomorphism). Manifolds M and N are *diffeomorphic* if there is 1-1 surjective map $f\colon M \to N$ such that both f and f^{-1} are smooth maps.

The following result is important as an illustration.

PROPOSITION 2.2.6. *The tangent bundle TS^1 of the circle S^1 is a 2-dimensional manifold diffeomorphic to an (infinite) cylinder $S^1 \times \mathbb{R}$.*

PROOF. We represent the circle as the set of complex numbers of unit length:

$$S^1 = \{e^{i\theta}\} \subseteq \mathbb{C}.$$

Then

(1) The vector field $\frac{d}{d\theta}$ is nonvanishing at every point of the circle.
(2) $e^{i(\theta+2\pi n)} = e^{i\theta}$ for all $n \in \mathbb{Z}$.
(3) At a point $e^{i\theta} \in S^1$, a tangent vector can be written as $t\frac{d}{d\theta}$ where $t \in \mathbb{R}$.

Such a tangent vector is uniquely determined by the real parameter t. Then the pair

$$(e^{i\theta}, t)$$

gives a parametrisation for the tangent bundle of S^1 by the Cartesian product $S^1 \times \mathbb{R}$, with $e^{i\theta} \in S^1$ and $t \in \mathbb{R}$. □

Remark 2.2.7 (Nontriviality of tangent bundle of the 2-sphere). Unlike the case of the circle, the tangent bundle of S^2 is not diffeomorphic to $S^2 \times \mathbb{R}^2$.[1] On the other hand, S^3 has a natural structure of a Lie group and is therefore parallelizable.[2] The sphere S^7 is parallelizable.[3]

Definition 2.2.8 (Canonical projection). Given the tangent bundle TM of a manifold M, let

$$\pi_M \colon TM \to M, \quad (p, v) \mapsto p \tag{2.2.2}$$

be the canonical projection "forgetting" the tangent vector v and keeping only its initial point p.

Remark 2.2.9. There is no "canonical projection" to the second component v.

2.3. Vector field as a section of the tangent bundle

Recall that we have a canonical projection $\pi_M \colon TM \to M$ of (2.2.2).

Definition 2.3.1. [Section] A vector field X on M is a *section*[4] of the tangent bundle. Namely, a vector field is a map $X \colon M \to TM$ satisfying the condition

$$\pi_M \circ X = \mathrm{Id}_M.$$

Thus a vector field on M is a rule that assigns, to each point of M, a tangent vector at that point. We will express a vector field more concretely in terms of local coordinates in Section 2.4.

2.4. Vector fields in coordinates

Consider a coordinate chart (A, u) in M where $u = (u^i)_{i=1,\ldots,n}$. When $p \in A$, we have a basis $(\frac{\partial}{\partial u^i})$ for T_pM by Theorem 2.1.1. Thus an arbitrary vector $X \in T_pM$ is a linear combination

$$X^i \frac{\partial}{\partial u^i},$$

for appropriate coefficients $X^i \in \mathbb{R}$ depending on the point p. Here we use the Einstein summation convention as usual.

---

[1] See the comments on the hairy ball theorem in Section 3.4.13, as well as Corollary 14.8.3.

[2] In fact, all orientable closed 3-manifolds are parallelizable; see e.g., `https://en.wikipedia.org/wiki/Parallelizable_manifold`

[3] Interesting details on the 7-dimensional case can be found at `https://mathoverflow.net/q/58131`.

[4] Chatach; mikta'

Recall that the vectors $\left(\frac{\partial}{\partial u^i}\right)$ form a basis for the tangent space at every point of a coordinate neighborhood $A \subseteq M$, by Theorem 2.1.1.

DEFINITION 2.4.1. A choice of component functions $X^i(u^1, \ldots, u^n)$ in the neighborhood defines a *vector field*

$$X^i(u^1, \ldots, u^n) \frac{\partial}{\partial u^i}$$

in the neighborhood A.

Here the components X^i are required to be of an appropriate differentiability type.[5] In more detail, we have the following definition.

DEFINITION 2.4.2. (see [**Boothby 1986**, p. 117]) Let M be a C^∞ manifold. A vector field X of class C^r on M is a map assigning to each point p of M, a vector $X_p \in T_pM$ whose components (X^i) in any local coordinate (A, u) are functions of class C^r.

2.5. Representing a vector by a path

EXAMPLE 2.5.1. Consider the derivation $\frac{\partial}{\partial x}$ at a point $p = (a, b) \in \mathbb{R}^2$. Then $\frac{\partial}{\partial x}$ can be represented by the path

$$\alpha(s) = (a + s, b),$$

in the sense that

$$(\forall f \in \mathbb{D}_p) \quad \frac{\partial}{\partial x} f = \tfrac{d}{ds}\Big|_{s=0} f(\alpha(s)).$$

EXAMPLE 2.5.2. Similarly, $\frac{\partial}{\partial y}$ at a point $p = (a, b)$ is represented by the path $\beta(s) = (a, b + s)$.

The representing path is not unique.

---

[5]Norms on tangent (and cotangent) bundles will be important in the sequel. We provide a preliminary illustration. Let M be the Euclidean plane $\mathbb{R}^2$. Via obvious identifications, the Euclidean norm in the (x, y)-plane leads naturally to a Euclidean norm $|\ |$ on the tangent space (i.e., tangent plane) at every point with respect to which both tangent vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are orthogonal and have unit norm: $\left|\frac{\partial}{\partial x}\right| = \left|\frac{\partial}{\partial y}\right| = 1$. Note that each of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ defines a global vector field on $\mathbb{R}^2$ (defined at every point of the plane). Any combination $X = X^1(x, y)\frac{\partial}{\partial x} + X^2(x, y)\frac{\partial}{\partial y}$ is also a vector field in the plane, with norm $|X| = \sqrt{(X^1)^2 + (X^2)^2}$ since the basis $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is orthonormal.

2.5.1. Vector fields in polar coordinates. The material in this subsection is optional.

REMARK 2.5.3 (Motivation for vector fields, differential forms, and de Rham cohomology). We will first discuss some examples of vector fields to motivate the introduction of differential forms starting in Section 4.1. Eventually we will develop the notion of a differential k-form, generalizing the notion of a 1-form. The 1-forms, also known as covectors, are dual to vectors. De Rham cohomology will be defined in terms of differential forms.

EXAMPLE 2.5.4. Significant examples of vector fields are provided by polar coordinates (r, θ). These may be undefined at the origin, i.e., a priori only defined in the open submanifold $\mathbb{R}^2 \setminus \{0\} \subseteq \mathbb{R}^2$. A point with polar coordinates (r, θ) appears in Cartesian coordinates as $(r\cos\theta, r\sin\theta)$.

Generalising the situation in Section 2.5, the vector $\frac{\partial}{\partial\theta}$ at such a point is represented by the path $\alpha(\theta) = (r\cos\theta, r\sin\theta)$ with derivative $\alpha'(\theta) = (-r\sin\theta, r\cos\theta) = r(-\sin\theta, \cos\theta)$, and therefore $|\alpha'| = r$. Hence at the point with polar coordinates (r, θ), we have

$$\left|\frac{\partial}{\partial\theta}\right| = r. \tag{2.5.1}$$

For an alternative argument see the note.[6]

COROLLARY 2.5.5. *The rescaled vector* $\frac{1}{r}\frac{\partial}{\partial\theta}$ *is of norm* 1.

2.5.2. Source, sink, circulation. The material in this subsection is optional

In this section we will describe some illustrative examples of vector fields and their integral curves.

EXAMPLE 2.5.6 (Zero of type source/sink). The vector field $\frac{\partial}{\partial r}$ in the plane is undefined at the origin, but the modifined vector field

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

has a smooth extension which is a vector field vanishing at the origin.

DEFINITION 2.5.7. The vector field in the plane defined by $r\frac{\partial}{\partial r}$ is a *source* while the opposite vector field $X = -r\frac{\partial}{\partial r}$ is a *sink*.[7]

[6]An alternative argument can be given in terms of differentials. Since $dr^2 = dx^2 + dy^2$ by Pythagoras, we have $|dr| = 1$ as well. Meanwhile $\theta = \arctan\frac{y}{x}$ and therefore $d\theta = \frac{1}{1+(y/x)^2}d(y/x) = \frac{x^2}{y^2+x^2}\frac{xdy-ydx}{x^2} = \frac{xdy-ydx}{r^2}$. Hence

$$|d\theta| = \frac{|xdy - ydx|}{r^2} = \frac{r}{r^2} = \frac{1}{r}. \tag{2.5.2}$$

Thus $rd\theta$ is a unit covector. We therefore have an orthonormal basis $(dr, rd\theta)$ for the cotangent space. Since $d\theta\left(\frac{\partial}{\partial\theta}\right) = 1$, equation (2.5.2) implies (2.5.1). Therefore $\frac{1}{r}\frac{\partial}{\partial\theta}$ is a unit vector.

[7]Makor and kior (with kaf) or rather *bor*.

REMARK 2.5.8 (Geometric description). The integral curves of a *source* flow from the origin and away from it, whereas the integral curves of a *sink* flow into the origin, and converge to it for large time. At a point $p \in \mathbb{R}^2$ with polar coordinates (r, θ), the sink is given by

$$X(p) = \begin{cases} -r\frac{\partial}{\partial r} & \text{if } p \neq 0 \\ 0 & \text{if } p = 0. \end{cases}$$

EXAMPLE 2.5.9 (Zero of type "circulation" in the plane). The vector $\frac{\partial}{\partial \theta}$ in the plane, viewed as a tangent vector at a point at distance r from the origin, tends to zero as r tends to 0, as is evident from (2.5.1). Therefore the vector field defined by $\frac{\partial}{\partial \theta}$ on $\mathbb{R}^2 \setminus \{0\}$ extends by continuity to the point $p = 0$. Moreover it can be expressed as

$$\frac{\partial}{\partial \theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

and hence smooth. Thus we obtain a continuous vector field $p \mapsto X(p) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$ which vanishes at the origin:

$$X(p) = \begin{cases} \frac{\partial}{\partial \theta} & \text{if } p \neq 0 \\ 0 & \text{if } p = 0. \end{cases}$$

Such a vector field is sometimes described as having *circulation*[8] around the point 0. The integral curves of a circulation are circles centered at the origin.

REMARK 2.5.10. The zero of the vector field associated with small oscillations of the pendulum[9] is of circulation type. These were studied in e.g., [**Kanovei et al. 2016**].

Next we discuss zeros of type *circulation* on the sphere.

LEMMA 2.5.11. *Consider the spherical coordinates (θ, φ) on S^2. The vector field $\frac{\partial}{\partial \theta}$ on the sphere has two zeros of circulation type, namely north and south poles.*

PROOF. Spherical coordinates (ρ, θ, φ) in $\mathbb{R}^3$ restrict to the unit sphere $S^2 \subseteq \mathbb{R}^3$ to give coordinates (θ, φ) on S^2. The north pole is defined by $\varphi = 0$. At this point, the angle θ is undefined but the vector field $\frac{\partial}{\partial \theta}$ can be extended by continuity as in the plane (see Example 2.5.9) (for a dual discussion see Section 7.14.1). Thus we obtain a zero of circulation type going counterclockwise. Similarly the south pole is defined by $\varphi = \pi$. Here $\frac{\partial}{\partial \theta}$ has a zero of circulation type but going clockwise with respect to the natural orientation on the 2-sphere. $\square$

[8] machzor, tzirkulatsia.

[9] Tnudot ktanot shel metutelet

2.6. Duality in linear algebra

We will deal with several notions of duality. The first one is a duality in linear algebra. Duality in differential geometry will be discussed in Section 2.7. Let V be a real vector space.

EXAMPLE 2.6.1. Euclidean space $\mathbb{R}^n$ is a real vector space of dimension n.

EXAMPLE 2.6.2. The tangent plane T_pM of a regular surface M (see Definition 1.1.4) at a point $p \in M$ is a real vector space of dimension 2.

DEFINITION 2.6.3. A *linear form*, also called 1-*form*, ϕ on a vector space V is a linear functional from V to $\mathbb{R}$.

DEFINITION 2.6.4. The *dual* space of V, denoted V^*, is the space of all linear forms λ on V. Namely, $V^* = \{\lambda\colon \lambda \text{ is a 1-form on } V\}$.

Evaluating λ at an element $x \in V$ produces a scalar $\lambda(x) \in \mathbb{R}$.

DEFINITION 2.6.5. The *natural pairing*[10] between V and V^* is a linear map

$$\langle\ ,\ \rangle\colon V \times V^* \to \mathbb{R},$$

defined by setting $\langle x, y\rangle = y(x)$, for all $x \in V$ and $y \in V^*$.

THEOREM 2.6.6. *If V admits a basis of vectors $(x_i)_{i=1,\ldots,n}$, then V^* admits a unique basis, called the* dual basis (y_j), *satisfying*

$$\langle x_i, y_j\rangle = \delta_{ij}, \tag{2.6.1}$$

for all $i, j = 1, \ldots, n$, where δ_{ij} is the Kronecker delta function.

EXAMPLE 2.6.7. The vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ form a basis for the tangent plane T_pE of the Euclidean plane E at each point $p \in E$. The dual space is denoted T_p^*E and called the cotangent plane.

DEFINITION 2.6.8. The basis dual to $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is denoted (dx, dy). Thus (dx, dy) is a basis for the cotangent plane T_p^* at every point $p \in E$.

Polar coordinates will be dealt with in detail in Section 2.6.1. They provide helpful examples of vectors and 1-forms, as follows.

EXAMPLE 2.6.9. In polar coordinates (r, θ), we have a basis $(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta})$ for the tangent plane T_pE of the Euclidean plane E at each point $p \in E\backslash\{0\}$. The dual space T_p^* has a basis denoted $(dr, d\theta)$ dual to $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right)$.

[10] zivug tiv'i

EXAMPLE 2.6.10. In polar coordinates, the 1-form $r\,dr$ occurs frequently in calculus. This 1-form vanishes at the origin, and gets "bigger and bigger" as we get further away from the origin, as discussed in Section 2.6.1.

2.6.1. Polar, cylindrical, and spherical coordinates. The material in this subsection is optional.

Polar coordinates[11] (r, θ) satisfy $r^2 = x^2 + y^2$ and $x = r\cos\theta$, $y = r\sin\theta$. In $\mathbb{R}^2 \setminus \{0\}$, one way of defining the ranges for the variables is to require $r > 0$ and $\theta \in [0, 2\pi)$. It is shown in elementary calculus that the area of a region D in the plane in polar coordinates is calculated using the area element $dA = r\,dr\,d\theta$. Thus, the area is expressed by the integral $\text{area}(D) = \int_D dA = \iint r dr d\theta$.

Cylindrical coordinates in Euclidean 3-space are studied in vector calculus.

DEFINITION 2.6.11. Cylindrical coordinates (koordinatot gliliot) (r, θ, z) are a natural extension of the polar coordinates (r, θ) in the plane.

The volume of an open region D is calculated with respect to cylindrical coordinates using the volume element $dV = r\,dr\,d\theta\,dz$. Thus the volume of D can be expressed as follows: $\text{vol}(D) = \int_D dV = \iiint r dr\,d\theta\,dz$.

EXAMPLE 2.6.12. Find the volume of a right circular cone with height h and base a circle of radius b.

Spherical coordinates[12] (ρ, θ, φ) in Euclidean 3-space are studied in vector calculus.

DEFINITION 2.6.13. Spherical coordinates (ρ, θ, φ) are defined as follows. The coordinate ρ is the distance from the point to the origin, satisfying $\rho^2 = x^2 + y^2 + z^2$, or $\rho^2 = r^2 + z^2$, where $r^2 = x^2 + y^2$. If we project the point orthogonally to the (x, y)-plane, the polar coordinates of its image, (r, θ), satisfy $x = r\cos\theta$ and $y = r\sin\theta$. The coordinate φ of a point in $\mathbb{R}^3$ is the angle between the position vector of the point and the third basis vector $e_3 = (0, 0, 1)^t$ in 3-space. Thus $z = \rho\cos\varphi$ while $r = \rho\sin\varphi$.

REMARK 2.6.14. The ranges of the coordinates are often chosen as follows: $0 \le \rho$, while $0 \le \theta \le 2\pi$, and $0 \le \varphi \le \pi$ (note the different upper bounds for θ and φ).

Recall that the volume of a region $D \subseteq \mathbb{R}^3$ is calculated using a volume element of the form $dV = \rho^2 \sin\varphi \; d\rho\,d\theta\,d\varphi$, so that the volume of a region D is $\text{vol}(D) = \int_D dV = \iiint_D \rho^2 \sin\varphi \; d\rho\,d\theta\,d\varphi$.

EXAMPLE 2.6.15. Calculate the volume of the spherical shell between spheres of radius $\alpha > 0$ and $\beta \ge \alpha$.

[11] koordinatot koteviot

[12] koordinatot kaduriot

REMARK 2.6.16. Consider a sphere S_ρ of radius $\rho = \beta$. The area of a spherical region on S_ρ is calculated using the area element $dA_{S_\rho} = \beta^2 \sin\varphi \; d\theta \, d\varphi$.

Thus the area of a spherical region $D \subseteq S_\beta$ is given by the integral area$(D) = \int_D dA_{S_\rho} = \iint \beta^2 \sin\varphi \; d\theta \, d\varphi$.

EXAMPLE 2.6.17. Calculate the area of the spherical region on a sphere of radius β included in the first octant, (so that all three Cartesian coordinates are positive).

2.7. Cotangent space and cotangent bundle

Derivations were already discussed in Section 1.9. Recall that the tangent space T_pM at $p \in M$ is the space of derivations at p. Duality plays an important role in differential geometry.

DEFINITION 2.7.1. The vector space dual to the tangent space T_p is the *cotangent space*, and denoted T_p^*.

Thus an element of a tangent space is a vector, while an element of a cotangent space is called a 1-form, or a *covector*.

DEFINITION 2.7.2. As a set, the *cotangent bundle*, denoted T^*M, of an n-dimensional manifold M is the disjoint union of all cotangent spaces T_p^*M as p ranges through M, or in formulas: $T^*M = \bigsqcup_{p\in M} T_p^*M$.

We generalize Definition 2.6.8.

DEFINITION 2.7.3. The basis dual to the basis $(\frac{\partial}{\partial u^i})_{i=1,\dots,n}$ is denoted (du^i), $\quad i = 1, \dots, n$.

THEOREM 2.7.4. *The cotangent bundle T^*M of a smooth n-dimensional manifold M is a smooth manifold of dimension $2n$.*

The proof is similar to that of Theorem 2.2.2 and appears in Section 4.1.

Thus each du^i is by definition a 1-form on T_p, or a cotangent vector (covector for short). We are therefore working with dual bases $\left(\frac{\partial}{\partial u^i}\right)$ for vectors, and (du^j) for covectors. The pairing as in formula (2.6.1) in Theorem 2.6.6 gives

$$\left\langle \frac{\partial}{\partial u^i}, du^j \right\rangle = du^j\left(\frac{\partial}{\partial u^i}\right) = \delta_i^j, \tag{2.7.1}$$

where δ_i^j is the Kronecker delta: $\delta_i^j = 1$ if $i = j$ and $\delta_i^j = 0$ if $i \neq j$. Examples of 1-forms in the plane are dx, dy, dr, rdr, $d\theta$. In analogy with vector fields, we will define a differential 1-form as a section of the cotangent bundle T^*M; see Chapter 4.

CHAPTER 3

Metric differential geometry

In chapters 1 and 2, we dealt mainly with differentiable manifolds M without additional structure, other than the existence of a distance function to ensure metrizability; see Section 1.4.

In the present chapter we will deal more systematically with *lengths* and *distances* on M defined via the first fundamental forms and Riemannian metrics.

This prepares the ground for Gromov's systolic inequality for complex projective space; see Section 9.14.[1]

3.1. Isometries, constructing bilinear forms out of 1-forms

DEFINITION 3.1.1. A *Riemannian metric* at a point $p \in M$ is a symmetric, positive definite, bilinear form on the tangent space T_p. A smooth assignment of a form at every point of M results in a globally defined metric on M.

A more detailed definition appears in Section 3.2.

DEFINITION 3.1.2. A *Riemannian manifold* is a differentiable manifold equipped with a metric.

A metric enables us to measure the length of paths in M. This is done by integrating the norm of the tangent vector of a parametrisation of the path.

DEFINITION 3.1.3. An *isometry* is a map f between Riemannian manifolds M and N which preserves the length of all paths. Thus, if $\gamma\colon [0,1] \to M$ is a smooth path in M then $|\gamma| = |f \circ \gamma|$, where absolute values denote the length.

DEFINITION 3.1.4. Manifolds M and N are *isometric* if there exists a one-to-one, onto isometry between them.

PROPOSITION 3.1.5. *To prescribe a metric it is sufficient to prescribe the corresponding quadratic form.*

[1]See also Loewner's systolic inequality for the torus in Chapter 12, and Pu's systolic inequality for the real projective plane in Chapter 13.

Proof. The polarisation formula allows one to reconstruct a symmetric bilinear form B, from the quadratic form $Q(v) = B(v,v)$, at least if the characteristic is not 2:

$$B(v,w) = \frac{1}{4}(Q(v+w) - Q(v-w)). \tag{3.1.1}$$

Thus the bilinear form can be recovered from the quadratic form. □

One can construct quadratic forms out of the 1-forms du^i on M. Namely a quadratic form is a linear combination of the rank-1 quadratic forms $(du^i)^2$.

Proposition 3.1.6. *A positive definite quadratic form (over the field of scalars $\mathbb{R}$) can be represented as a sum $a_i(du^i)^2$ where $a_i \in \mathbb{R}^+$ and (du^i) is the basis dual to $\left(\frac{\partial}{\partial u^i}\right)$.*

Proof. The proof is immediate from orthogonal diagonalisation of symmetric matrices.[2] □

Each bilinear form on the tangent space T_pM is the polarisation of a suitable quadratic form (3.1.1).

3.2. Riemannian metric, first fundamental form

Let M be a manifold, and $p \in M$ a point on the manifold. Recall that the tangent space T_p at a point p is the fiber[3] (i.e., inverse image of a point) of the projection map $\pi_M \colon TM \to M$.

Definition 3.2.1. A *Riemannian metric* $\mathbf{g}$ on M is choice of a symmetric positive definite bilinear form on the tangent space $T_p = T_pM$ at p:

$$\mathbf{g} \colon T_p \times T_p \to \mathbb{R},$$

defined for all $p \in M$ and varying smoothly as a function of p.

Example 3.2.2. The more naive viewpoint[4] is as follows. Here we use the terminology *first fundamental form* instead of *Riemannian metric*. Consider an open set $U \subseteq \mathbb{R}^2$, with coordinates (u^1, u^2). Assume U is embedded in Euclidean 3-space $\mathbb{R}^3$ by means of a regular[5] map $\underline{x}(u^1, u^2)$,

$$\underline{x} \colon U \to \mathbb{R}^3 .$$

[2] See `http://u.math.biu.ac.il/~katzmik/88-201.html` for details.

[3] siv

[4] Such a viewpoint is taken in the course 88201; see `http://u.math.biu.ac.il/~katzmik/88-201.html`

[5] I.e., the Jacobian $J_{\underline{x}}$ is of rank 2.

The image of $\underline{x}$ is a surface $M \subseteq \mathbb{R}^3$. The tangent plane of the resulting surface M in $\mathbb{R}^3$ is spanned by vectors

$$x_i = \frac{\partial \underline{x}}{\partial u^i} \quad \text{where} \quad i = 1, 2. \tag{3.2.1}$$

The ambient space $\mathbb{R}^3$ is equipped with a standard inner product denoted $\langle\, , \rangle_{\mathbb{R}^3}$. The restriction of the inner product to the tangent plane of M gives a first fundamental form on M:

$$\mathbf{g}\colon T_p \times T_p \to \mathbb{R}, \quad \mathbf{g}(v, w) = \langle v, w\rangle_{\mathbb{R}^3}.$$

The first fundamental form is traditionally expressed by a matrix of coefficients called *metric coefficients* g_{ij}, with respect to coordinates (u^1, u^2) on the surface.

Definition 3.2.3. The *metric coefficient* g_{ij} of a surface $M \subseteq \mathbb{R}^3$ parametrized by $\underline{x}$ is given by the inner product of the i-th and the j-th vector of the basis of T_pM:

$$g_{ij} = \left\langle \frac{\partial x}{\partial u^i}, \frac{\partial x}{\partial u^j} \right\rangle_{\mathbb{R}^3}. \tag{3.2.2}$$

In particular, the diagonal coefficient satisfies $g_{ii} = \left|\frac{\partial x}{\partial u^i}\right|^2$, namely it is the square length of the i-th basis vector $\frac{\partial x}{\partial u^i}$.

3.3. Metric as sum of squared 1-forms; element of length ds

In this section, we go beyond the naive viewpoint summarized in Section 3.2 and adopt the modern viewpoint developed in the previous chapter.

In the case of surfaces, at every point $p = (u^1, u^2)$, we have the metric coefficients $g_{ij} = g_{ij}(u^1, u^2)$. Each metric coefficient is thus a scalar-valued function of two variables.

Consider the case when the matrix (g_{ij}) is diagonal. This can always be achieved *at a point* p by a change of coordinates in a neighborhood of p (see Proposition 3.1.6). Suppose this is true everywhere in a coordinate chart.[6] Then in the notation developed in Section 3.1, we can write the Riemannian metric as follows:

$$\mathbf{g} = g_{11}(u^1, u^2)(du^1)^2 + g_{22}(u^1, u^2)(du^2)^2. \tag{3.3.1}$$

[6]Such a property is weaker than having isothermal coordinates where one requires the matrix to be a scalar matrix at every point (possibly depending on the point); see Section 3.4.5.

EXAMPLE 3.3.1. If the metric coefficients form an identity matrix, we obtain the *standard flat metric*

$$\mathbf{g} = \left(du^1\right)^2 + \left(du^2\right)^2. \tag{3.3.2}$$

Here the interior superscript (inside the parentheses) denotes an index, while exterior superscript denotes the squaring operation.

Sometimes it is convenient to simplify notation by replacing the coordinates (u^1, u^2) simply by (x, y). Then the standard flat metric (3.3.2) is $\mathbf{g} = dx^2 + dy^2$. Often the metric is expressed in terms of the length element ds by writing

$$ds^2 = dx^2 + dy^2, \tag{3.3.3}$$

or, more generally, $ds^2 = g_{11}(x, y)\, dx^2 + g_{22}(x, y)\, dy^2$.

3.4. Hyperbolic metric; lattices

The hyperbolic metric $\mathbf{g}_{hyp}$ in the upperhalf plane $\{(x, y)\colon y > 0\}$ is the Riemannian metric expressed by the quadratic form

$$\mathbf{g}_{hyp} = \frac{1}{y^2}(dx^2 + dy^2), \quad \text{where} \quad y > 0. \tag{3.4.1}$$

Thus its length element ds satisfies $ds = \frac{1}{y}\sqrt{dx^2 + dy^2}$. Note that this expression is undefined for $y = 0$. The hyperbolic metric in the upper half plane is a complete metric.

THEOREM 3.4.1. *The Gaussian curvature $K = K(x, y)$ of the metric* (3.4.1) *satisfies $K = -1$ at every point.*

PROOF. We use the formula for for the Gaussian curvature involving the Laplace-Beltrami operator $\Delta_{LB} = \frac{1}{f^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ for the metric $f^2(dx^2 + dy^2)$. Namely, we have

$$K = -\Delta_{LB} \ln f \quad \text{for the metric} \quad f^2(dx^2 + dy^2), \tag{3.4.2}$$

and use the fact that the second derivative of $\ln y$ is $-\frac{1}{y^2}$.[7] □

It is a deep theorem that upperhalf plane with the hyperbolic metric does not admit a global isometric embedding in Euclidean space.[8]

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[7] For details see `http://u.math.biu.ac.il/~katzmik/egreglong.pdf`.

[8] In 1901, D. Hilbert [**9**] proved that there is no global isometric immersion of the hyperbolic plane into 3-dimensional Euclidean space, despite the fact that there is a local isometric embedding. See `http://davidbrander.org/penn.pdf`

3.4.1. Surface of revolution in $\mathbb{R}^3$. From here until the end of chapter 3, the material is optional.

In Section 3.3, we developed a formalism for describing a metric on an arbitrary surface, in coordinates (u^1, u^2). In the special case of a surface of revolution in $\mathbb{R}^3$, it is customary to use the notation $\begin{cases} u^1 = \theta \\ u^2 = \varphi \end{cases}$ for the coordinates, as in formula (3.4.4) below. Then the Riemannian metric $\mathbf{g}$ of (3.3.1) can then be written as[9] $\mathbf{g} = g_{11}(\theta, \varphi)\,(d\theta)^2 + g_{22}(\theta, \varphi)\,(d\varphi)^2$ which can be abbreviated as $\mathbf{g} = g_{11}(\theta, \varphi)\,d\theta^2 + g_{22}(\theta, \varphi)\,d\varphi^2$. The starting point in the construction of a surface of revolution in $\mathbb{R}^3$ is an embedded regular curve C in the xz-plane, called the *generating curve*. The curve C is parametrized by a pair of functions

$$\begin{cases} x = f(\varphi), \\ z = g(\varphi). \end{cases} \tag{3.4.3}$$

We will assume that $f(\varphi) > 0$ for all φ.

Definition 3.4.2. The *surface of revolution* (around the z-axis), generated by the curve C, is the surface parametrized as follows:

$$\underline{x}(\theta, \varphi) = (f(\varphi)\cos\theta, f(\varphi)\sin\theta, g(\varphi)). \tag{3.4.4}$$

The condition $f(\varphi) > 0$ ensures that the resulting surface is embedded in $\mathbb{R}^3$.

Remark 3.4.3. If the generating curve $C \subseteq \mathbb{R}^2$ is a Jordan curve in the plane (and, as before, $f(\varphi) > 0$) then the resulting surface is an embedded *torus*; see Section 3.4.6.

Definition 3.4.4. A pair of functions $(f(\varphi), g(\varphi))$ provides an *arclength parametrisation* of the curve C if

$$\forall\varphi,\ \left(\tfrac{df}{d\varphi}\right)^2 + \left(\tfrac{dg}{d\varphi}\right)^2 = 1. \tag{3.4.5}$$

Proposition 3.4.5. *Let $(f(\varphi), g(\varphi))$ be an arclength parametrisation of the generating curve. Then in the notation of Section* 3.3*, the metric* $\mathbf{g}$ *of the surface of revolution is given by the formula* $\mathbf{g} = f^2(\varphi)d\theta^2 + d\varphi^2$.

Proof of Proposition 3.4.5. To calculate the first fundamental form of the surface of revolution (3.4.4), note that the tangent vectors (see (3.2.1)) are $x_1 = \frac{\partial x}{\partial\theta} = (-f\sin\theta, f\cos\theta, 0)^t$, while $x_2 = \frac{\partial x}{\partial\varphi} = \left(\frac{df}{d\varphi}\cos\theta, \frac{df}{d\varphi}\sin\theta, \frac{dg}{d\varphi}\right)^t$. Thus we have $g_{11} = f^2\sin^2\theta + f^2\cos^2\theta = f^2$ and

$$g_{22} = \left(\tfrac{df}{d\varphi}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\tfrac{dg}{d\varphi}\right)^2 = \left(\tfrac{df}{d\varphi}\right)^2 + \left(\tfrac{dg}{d\varphi}\right)^2.$$

[9]We use the boldface (mudgash) font for $\mathbf{g}$ of the metric mainly to distinguish it from the function g occurring below in (3.4.3).

Furthermore, $g_{12} = -f\frac{df}{d\varphi}\sin\theta\cos\theta + f\frac{df}{d\varphi}\cos\theta\sin\theta = 0$. Thus the Riemannian metric or the first fundamental form is

$$(g_{ij}) = \begin{pmatrix} f^2 & 0 \\ 0 & \left(\frac{df}{d\varphi}\right)^2 + \left(\frac{dg}{d\varphi}\right)^2 \end{pmatrix}. \tag{3.4.6}$$

If φ is an arclength parameter then $g_{22} = 1$ and the proposition follows from (3.4.5). □

Using 1-forms in place of matrix notation, equation (3.4.6) can be reformulated as follows.

Corollary 3.4.6. *Assume $(f(\varphi), g(\varphi))$ is a regular parametrisation of the generating curve. In the notation of Section* 3.3*, the Riemannian metric* $\mathbf{g}$ *of the surface of revolution is given by the formula*

$$\mathbf{g} = f^2 d\theta^2 + \left(\left(\frac{df}{d\varphi}\right)^2 + \left(\frac{dg}{d\varphi}\right)^2\right) d\varphi^2.$$

Example 3.4.7. The unit sphere is a surface of revolution whose generating curve C is the circle. Setting $f(\varphi) = \sin\varphi$ and $g(\varphi) = \cos\varphi$, we obtain a parametrisation of the sphere S^2 in spherical coordinates, with respect to which the metric takes the form

$$\mathbf{g}_{S^2} = \sin^2\varphi \, d\theta^2 + d\varphi^2. \tag{3.4.7}$$

In other words, $g_{11} = \sin^2\varphi$ and $g_{22} = 1$. The corresponding element of area is therefore $\sin\varphi \, d\theta \, d\varphi$.

Remark 3.4.8. As in (3.4.7), in general the first fundamental form of a surface of revolution as parametrized above is diagonal but not necessarily scalar. In Section 3.4.5, we will perform an appropriate change of coordinates so as to express the metric of a surface of revolution by a first fundamental form given by a *scalar* matrix (where the value of the scalar is a function of the point of the surface).

3.4.2. Coordinate change. In this section we will compare different coordinate charts for a surface $M \subseteq \mathbb{R}^3$. Given a metric in one coordinate chart, we would like to understand how the metric coefficients tranform under change of coordinates. Consider coordinate charts $(A, (u^i))$ and $(B, (v^\alpha))$. Consider a change from a coordinate chart $(u^i), \quad i = 1, 2$ to the coordinate chart $(v^\alpha), \quad \alpha = 1, 2$. In the overlap $A \cap B$ of the two domains, the coordinates can be expressed in terms of each other, e.g., $v = v(u)$.

Definition 3.4.9. We will use the following notation:

- $g_{ij} = g_{ij}(u^1, u^2)$ for the metric coefficients of M with respect to chart (u^i);
- $\tilde{g}_{\alpha\beta} = \tilde{g}_{\alpha\beta}(v^1, v^2)$ for the metric coefficients with respect to the chart (v^α).

We will use the Einstein summation convention; see Definition 2.2.3.

PROPOSITION 3.4.10. *Consider a surface $M \subseteq \mathbb{R}^3$. Under a coordinate change, the metric coefficients transform as follows:* $\tilde{g}_{\alpha\beta} = g_{ij} \frac{\partial u^i}{\partial v^\alpha} \frac{\partial u^j}{\partial v^\beta}$.

PROOF. By assumption, the metric in a neighborhood $U \subseteq M$ is induced by a Euclidean embedding $\underline{x}\colon U \to \mathbb{R}^3$, defined by $\underline{x} = \underline{x}(u) = \underline{x}(u^1, u^2)$. Changing coordinates, we obtain a new parametrisation $\underline{y}(v) = \underline{x}(u(v))$. Applying chain rule and bilinearity of inner product to (3.2.2), we obtain

$$\begin{aligned}
\tilde{g}_{\alpha\beta} &= \left\langle \frac{\partial y}{\partial v^\alpha}, \frac{\partial y}{\partial v^\beta} \right\rangle \\
&= \left\langle \frac{\partial x}{\partial u^i} \frac{\partial u^i}{\partial v^\alpha}, \frac{\partial x}{\partial u^j} \frac{\partial u^j}{\partial v^\beta} \right\rangle \\
&= \frac{\partial u^i}{\partial v^\alpha} \frac{\partial u^j}{\partial v^\beta} \left\langle \frac{\partial x}{\partial u^i}, \frac{\partial x}{\partial u^j} \right\rangle \\
&= g_{ij} \frac{\partial u^i}{\partial v^\alpha} \frac{\partial u^j}{\partial v^\beta},
\end{aligned}$$

completing the proof. □

3.4.3. Conformal equivalence.

DEFINITION 3.4.11. Two metrics, $\mathbf{g} = g_{ij} du^i du^j$ and $\mathbf{h} = h_{ij} du^i du^j$, on a surface M are *conformally equivalent, or* conformal *for short, if there exists a function* $f = f(u^1, u^2) > 0$ *such that* $\mathbf{g} = f^2 \mathbf{h}$*; in other words,*

$$g_{ij} = f^2 h_{ij} \quad \textit{for all} \quad i, j. \tag{3.4.8}$$

DEFINITION 3.4.12. The function f above is called the *conformal factor* of the metric $\mathbf{g}$ with respect to the metric $\mathbf{h}$.

REMARK 3.4.13. Sometimes the function $\lambda = f^2$ is referred to as the conformal factor.

EXAMPLE 3.4.14. The hyperbolic metric $\frac{1}{y^2}(dx^2 + dy^2)$ of Section 3.4 is conformally equivalent to the flat one $(dx^2 + dy^2)$, with conformal factor $f(x, y) = \frac{1}{y}$.

LEMMA 3.4.15. *Under a conformal change of metric with conformal factor f, the length of every vector at a point $p = (u^1, u^2)$ is multiplied by the factor $f(u^1, u^2)$.*

PROOF. Consider a vector $v = v^i \frac{\partial}{\partial u^i}$ at a point p which is a unit vector for the metric $\mathbf{h}$. Let us show that v is "stretched" by a factor of $f(p)$. In other words, its length with respect to $\mathbf{g}$ equals $f(p)$. Indeed, the new length of v is

$$\begin{aligned}\sqrt{\mathbf{g}(v,v)} &= \mathbf{g}\left(v^i\frac{\partial}{\partial u^i}, v^j\frac{\partial}{\partial u^j}\right)^{\frac{1}{2}} \\ &= \left(\mathbf{g}\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)v^iv^j\right)^{\frac{1}{2}} \\ &= \sqrt{g_{ij}v^iv^j} \\ &= \sqrt{f^2(p)h_{ij}v^iv^j} \\ &= f(p)\sqrt{h_{ij}v^iv^j} \\ &= f(p).\end{aligned}$$

□

DEFINITION 3.4.16. A *conformal structure*[10] *on M is an equivalence class of metrics on a surface M conformal to each other.*

3.4.4. Lattices, uniformisation theorem for tori. The following result is a consequence of the uniformisation theorem.

THEOREM 3.4.17. *Locally, every metric on a surface can be written as $f(x,y)^2(dx^2+dy^2)$ with respect to suitable coordinates (x,y).*

There is a global version of this theorem for tori, namely the uniformisation theorem in the genus 1 case (i.e., for tori); see Theorem 3.4.24.

Recall that by the Pythagorian theorem, the square-length of a vector in the plane equipped with the standard metric is the sum of the squares of its components, or

$$ds^2 = dx^2 + dy^2. \tag{3.4.9}$$

We will use the notation ds^2 of (3.4.9) as shorthand for this standard flat metric, as in (3.4.2).

DEFINITION 3.4.18 (Lattice). A *lattice* $L \subseteq \mathbb{R}^2$ a subgroup isomorphic to $\mathbb{Z}^2$ which is not included in any line in $\mathbb{R}^2$.

EXAMPLE 3.4.19. The Gaussian integers $L_G \subseteq \mathbb{R}^2$ is the lattice consisting of points with integer coordinates: $L_G = \{(m,n) \in \mathbb{R}^2 : m,n \in \mathbb{Z}\}$.

DEFINITION 3.4.20. Let L be a lattice in the plane. A function $f(p)$ on $\mathbb{R}^2$ is called L-periodic if $f(p+\ell) = f(p)$ for all $\ell \in L$ and all $p = (x,y) \in \mathbb{R}^2$.

DEFINITION 3.4.21. A *flat torus* is a quotient $\mathbb{R}^2/L$, where L is a lattice in the plane.

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REMARK 3.4.22. Note that the constant 1-forms dx and dy in the plane are in particular invariant under the translations $\mathbb{R}^2 \to \mathbb{R}^2$, $p \mapsto p + \ell$ where $\ell \in L$. This enables us to view dx and dy as 1-forms on the torus $\mathbb{R}^2/L$, as well.

EXAMPLE 3.4.23. Let $c, d \in \mathbb{R} \setminus \{0\}$. A rectangular lattice $L_{c,d} \subseteq \mathbb{R}^2$ is defined by $L_{c,d} = \{cme_1 + dne_2 \colon n, m \in \mathbb{Z}\}$. One can also view $L_{c,d}$ as a subset of $\mathbb{C}$ given by $L_{c,d} = \{cm + dni \colon n, m \in \mathbb{Z}\}$.

A more detailed version of the Uniformisation Theorem 3.4.17 in the case of tori characterizes Riemannian tori up to isometry (see Definition 3.1.4) in terms of the corresponding lattice, as follows.

THEOREM 3.4.24 (Uniformisation theorem for tori). *For every metric* $\mathbf{g}$ *on the* 2*-torus* $\mathbb{T}^2$*, there exists a lattice* $L \subseteq \mathbb{R}^2$ *and a positive* L*-periodic function* $f(x, y)$ *on* $\mathbb{R}^2$ *such that the torus* $(\mathbb{T}^2, \mathbf{g})$ *is isometric to*

$$\left(\mathbb{R}^2/L, f^2 ds^2\right), \tag{3.4.10}$$

where $ds^2 = dx^2 + dy^2$ *is the standard flat metric of* $\mathbb{R}^2$*.*[11]

3.4.5. Isothermal coordinates.

DEFINITION 3.4.25. Coordinates (u^1, u^2) in a neighborhood $U \subseteq M$ are called *isothermal* if the matrix (g_{ij}) of the first fundamental form (metric) of M, with respect to (u^1, u^2), is a scalar matrix at every point of U.

In terms of the metric coefficients g_{ij}, the isothermal condition is expressed by the pair of equations $g_{11} = g_{22}$ and $g_{12} = 0$. The following result expresses the metric of a surface of revolution in isothermal coordinates.

Suppose $(f(\varphi), g(\varphi))$, where $f(\varphi) > 0$, is an arclength parametrisation of the generating curve[12] of a surface of revolution in $\mathbb{R}^3$. Consider the change of variable

$$\psi = \int \frac{1}{f(\varphi)} d\varphi. \tag{3.4.11}$$

[11]More generally, there is also a global form of the theorem in arbitrary genus, which can be stated in several ways. One of such ways involves Gaussian curvature: *Every metric on a connected (kashir) surface is conformally equivalent to a metric of constant Gaussian curvature.* From the complex analytic viewpoint, the uniformisation theorem states that every Riemann surface is covered by either the sphere, the plane, or the upper halfplane. Thus no notion of curvature is needed for the statement of the uniformisation theorem. However, from the differential geometric point of view, what is relevant is that every conformal class of metrics contains a metric of constant Gaussian curvature. See [**Ab81**] for a lively account of the history of the uniformisation theorem. More information on the uniformisation theorem and the Riemann mapping theorem can be found at `https://mathoverflow.net/q/10516`.

[12]This curve does not have to be closed. We will specialize to Jordan curves in Section 3.4.6.

THEOREM 3.4.26. *With respect to the new parametrisation of the surface in terms of variables* (θ, ψ) *with* ψ *as in* (3.4.11)*, the metric becomes* $f^2(\varphi(\psi))(d\theta^2 + d\psi^2)$*, so that the matrix of metric coefficients is the scalar matrix* $(f^2\delta_{ij})$*.*

A proof is given below following Corollary 3.4.27.

COROLLARY 3.4.27. *The change of variables* (3.4.11) *produces an explicit conformal equivalence between the metric on the surface of revolution and the standard flat metric* $d\theta^2 + d\psi^2$ *in the* (θ, ψ) *coordinates.*

The existence of such a parametrisation is predicted by the uniformisation theorem (see Theorem 3.4.24) in the case of a general regular surface. The advantage of formula (3.4.11) is its explicit nature.

PROOF OF THEOREM 3.4.26. Let $\varphi = \varphi(\psi)$. We can assume that the dependence of φ on ψ is monotone. Then there exists an inverse function $\psi = \psi(\varphi)$. By chain rule, $\frac{df}{d\psi} = \frac{df}{d\varphi}\frac{d\varphi}{d\psi}$. Now consider again the first fundamental form of Proposition 3.4.6. To impose the condition

$$g_{11} = g_{22}, \tag{3.4.12}$$

we need to solve the equation $f^2 = \left(\frac{df}{d\psi}\right)^2 + \left(\frac{dg}{d\psi}\right)^2$, or

$$f^2 = \left(\left(\frac{df}{d\varphi}\right)^2 + \left(\frac{dg}{d\varphi}\right)^2\right)\left(\frac{d\varphi}{d\psi}\right)^2.$$

In the case when the generating curve is parametrized by arclength, we are therefore reduced to the equation $f = \frac{d\varphi}{d\psi}$. Equivalently, we have $\psi = \int \frac{d\varphi}{f(\varphi)}$. Thus we obtain a parametrisation of the surface of revolution in coordinates (θ, ψ), such that the matrix of metric coefficients satisfies the relation (3.4.12) and is therefore a scalar matrix. $\square$

3.4.6. Tori of revolution. If the generating curve $C \subseteq \mathbb{R}^2$ is a Jordan curve and $f(\varphi) > 0$ then the resulting surface is a torus of revolution. Assuming that φ is the arclength paramenter, the change of variables $\psi = \int \frac{d\varphi}{f(\varphi)}$ results in isothermal coordinates (θ, ψ) on the torus, by Theorem 3.4.26. The lattice L of such a torus in the (θ, ψ)-plane is rectangular (see Example 3.4.23). The following is immediate from the results of the previous section.

COROLLARY 3.4.28. *Suppose* C *is a Jordan curve in the plane. For the isothermal* (θ, ψ)*-parametrisation of the torus of revolution with generating curve* C*, the variable* θ *ranges from* 0 *to* 2π*, while the variable* ψ *varies from* 0 *to* $\int_0^\mu \frac{d\varphi}{f(\varphi)}$ *where* μ *is the length of the curve* C*.*

In more detail, the result of Section 3.4.5 has the following corollary.

Corollary 3.4.29. *Consider a torus of revolution* $(\mathbb{T}^2, \mathbf{g}_{rev})$ *in* $\mathbb{R}^3$ *formed by rotating a Jordan curve of length* $\mu > 0$*, with unit speed parametisation* $(f(\varphi), g(\varphi))$ *where* $\varphi \in [0, \mu]$. *Then the torus is conformally equivalent to a flat torus* $\mathbb{R}^2/L_{c,d}$*, defined by a rectangular lattice* $L_{c,d} \subseteq \mathbb{R}^2$*, where* $\begin{cases} c = 2\pi \\ d = \int_0^\mu \frac{d\varphi}{f(\varphi)}. \end{cases}$ *The metric in coordinates* (θ, ψ) *is given by* $\mathbf{g}_{rev} = f^2(\varphi(\psi))(d\theta^2 + d\psi^2)$*, for the change of coordinate* $\psi = \int \frac{d\varphi}{f(\varphi)}$.

Figure 3.4.1. Torus: lattice (left) and embedding (right)

3.4.7. Standard fundamental domain, conformal parameter τ. In this section we undertake a more detailed study of flat tori or equivalently, lattices in the plane. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of a lattice L as a subgroup of $\mathbb{C}$.

Definition 3.4.30. We say that lattice $L \subseteq \mathbb{C}$ is *similar* to lattice $L' \subseteq \mathbb{C}$ if there exists a number $\lambda \in \mathbb{C}$ such that $L = \lambda L'$.

Writing $\lambda = re^{i\theta}$ we see that similar lattices differ by a rotation and multiplication by a real scalar. Similarly, we will say that the corresponding flat tori $\mathbb{C}/L$ and $\mathbb{C}/L'$ are similar. The standard fundamental domain in the complex plane is important in many branches of mathematics, and is defined as follows.

Definition 3.4.31. The *standard fundamental domain* $D_0 \subseteq \mathbb{C}$ is the region

$$D_0 = \{z = x + iy \in \mathbb{C}\colon\ -\tfrac{1}{2} \leq x < \tfrac{1}{2},\ y > 0,\ |z| \geq 1\}. \tag{3.4.13}$$

Theorem 3.4.32. *Every flat torus* $\mathbb{C}/L$ *is similar to the torus obtained as the quotient of* $\mathbb{C}$ *by the lattice spanned by the pair* $(1, \tau)$ *where* $\tau \in D_0$ *as in* (3.4.13). *The parameter* τ *is called the* conformal parameter[13] *of the torus.*

[13]Not to be confused with *conformal factor*.

A proof appears in Section 10.4 in the context of a proof of Loewner's systolic inequality in Chapter 12.

3.4.8. Conformal parameter of rectangular lattices. Consider the rectangular lattice $L_{c,d} \subseteq \mathbb{C}$ where $c > 0$ and $d > 0$ (see Example 3.4.23). Thus $L_{c,d} = \mathrm{Span}_{\mathbb{Z}}(c\,1, d\,i)$.

LEMMA 3.4.33. *The conformal parameter τ of the lattice $L_{c,d}$ is $\tau(L_{c,d}) = i \max\left(\frac{c}{d}, \frac{d}{c}\right)$.*

PROOF. Suppose $d \geq c$. Then we scale the lattice by a factor of $\frac{1}{c}$ to obtain a similar lattice

$$L' = L'_{1,\frac{d}{c}} = \mathrm{Span}_{\mathbb{Z}}(1, \tfrac{d}{c} i).$$

Since $\frac{d}{c} \geq 1$ we have $\frac{d}{c} i \in D_0$ and therefore the conformal parameter of L' is $\tau = \frac{d}{c} i$.

Now suppose $c \geq d$. Then we scale the lattice by a factor of $\frac{1}{d}$ and also multiply by $e^{i\frac{\pi}{2}}$ (rotation by a right angle). We then obtain a similar lattice

$$L' = L'_{1,\frac{c}{d}} = \mathrm{Span}_{\mathbb{Z}}(1, \tfrac{c}{d} i).$$

In this case the conformal parameter is $\tau = \frac{c}{d} i \in D_0$. □

In Section 3.4.6 we studied the torus of revolution generated by a Jordan curve of length μ with arclength parametrisation $(f(\varphi), g(\varphi))$. If a lattice is rectangular, the conformal parameter τ is pure imaginary. We therefore obtain the following corollary.

COROLLARY 3.4.34. *The conformal parameter τ of a torus of revolution is pure imaginary: $\tau = iy_0$ of absolute value $y_0 = \max\left(\frac{c}{d}\,,\ \frac{d}{c}\right) \geq 1$, where $c = 2\pi$ and $d = \int_0^\mu \frac{d\varphi}{f(\varphi)}$, where μ is the length of the generating Jordan curve.*

3.4.9. Conformal parameter of standard tori of revolution. We will use the expression in terms of an integral obtained in Section 3.4.8 to compute the conformal parameter τ of the standard tori embedded in $\mathbb{R}^3$.

DEFINITION 3.4.35. Let $a, b \in \mathbb{R}$ such that $0 < b < a$. The torus of revolution $T_{a,b} \subseteq \mathbb{R}^3$ is defined by the generating curve C in the (x, z) plane:

$$C = \left\{(x, z) \in \mathbb{R}^2 \colon (x-a)^2 + z^2 = b^2\right\}. \tag{3.4.14}$$

By Corollary 3.4.34, the torus $T_{a,b}$ is equivalent to a flat torus with pure imaginary conformal parameter τ. We wish to specify the flat structure in the conformal class of the torus. We choose an arclength parametrisation of the generating curve C:

$$f(\varphi) = a + b\cos\frac{\varphi}{b}, \quad g(\varphi) = b\sin\frac{\varphi}{b}, \tag{3.4.15}$$

where $\varphi \in [0, \mu]$ with $\mu = 2\pi b$, and $b < a$. Recall that $\psi = \int \frac{d\phi}{f(\phi)}$.

THEOREM 3.4.36. *The flat torus in the conformal class of the torus of revolution $T_{a,b}$ is given by a rectangular lattice in the (θ, ψ) plane of the form $L_{c,d}$ where*

$$\begin{cases} c = 2\pi \\ d = \frac{2\pi}{\sqrt{(a/b)^2-1}}. \end{cases}$$

The conformal parameter τ of the torus $T_{a,b}$ is

$$\tau = i \max\left(\frac{1}{\sqrt{(a/b)^2-1}}, \sqrt{(a/b)^2-1}\right).$$

PROOF. As in Corollary 3.4.34, we replace φ by $\varphi(\psi)$ to produce isothermal coordinates (θ, ψ) for the standard torus of revolution parametrized as in (3.4.15), where

$$\psi = \int \frac{d\varphi}{f(\varphi)} = \int \frac{d\varphi}{a + b\cos\frac{\varphi}{b}}.$$

Therefore the flat metric is defined by a lattice in the (θ, ψ) plane with $c = 2\pi$ and

$$d = \int_0^{\mu=2\pi b} \frac{d\varphi}{a + b\cos\frac{\varphi}{b}}. \tag{3.4.16}$$

To evaluate the integral (3.4.16), we change the variable to $t = \frac{\varphi}{b}$ and let $\alpha = \frac{a}{b} > 1$. Then

$$d = \int_0^{2\pi} \frac{dt}{\alpha + \cos t}.$$

This is a standard integral whose value $\frac{2\pi}{\sqrt{\alpha^2-1}}$ is determined by a residue calculation in Lemma 3.4.37 below, proving the theorem. □

3.4.10. A residue calculation. The material in this subsection is optional.

LEMMA 3.4.37. *Let $\alpha > 1$. We have the following value of the definite integral: $\int_0^{2\pi} \frac{dt}{\alpha+\cos t} = \frac{2\pi}{\sqrt{\alpha^2-1}}$.*

PROOF. We will use the residue theorem for complex functions. First note that $\int_0^{2\pi} \frac{dt}{\alpha+\cos t} = \int_0^{2\pi} \frac{dt}{\alpha+\mathrm{Re}(e^{it})} = \int_0^{2\pi} \frac{2dt}{2\alpha+e^{it}+e^{-it}}$. The change of variables $z = e^{it}$ yields $dt = \frac{-idz}{z}$. We thus obtain the following integral along the unit circle:

$$\begin{aligned} \oint \frac{-2idz}{z(2\alpha + z + z^{-1})} &= \oint \frac{-2idz}{z^2 + 2\alpha z + 1} \\ &= \oint \frac{-2idz}{(z-\lambda_1)(z-\lambda_2)}, \end{aligned} \tag{3.4.17}$$

where

$$\lambda_1 = -\alpha + \sqrt{\alpha^2 - 1}, \quad \lambda_2 = -\alpha - \sqrt{\alpha^2 - 1}. \tag{3.4.18}$$

From (3.4.18) we obtain

$$\lambda_1 - \lambda_2 = 2\sqrt{\alpha^2 - 1}. \tag{3.4.19}$$

The root λ_2 is outside the unit circle. Thus $z = \lambda_1$ is the only singularity of the meromorphic function under the integral sign in (3.4.17) inside the unit circle. Hence we need to compute the residue at λ_1 so as to apply the residue theorem. The residue at λ_1 is obtained by multiplying through by $(z-\lambda_1)$ and then evaluating at $z = \lambda_1$. We therefore obtain from (3.4.19) that $\mathrm{Res}_{\lambda_1} = \frac{-2i}{\lambda_1-\lambda_2} = \frac{-2i}{2\sqrt{\alpha^2-1}} = \frac{-i}{\sqrt{\alpha^2-1}}$. The integral is determined by the residue theorem in terms of the residue at the pole $z = \lambda_1$. Therefore the integral equals $2\pi i\,\mathrm{Res}_{\lambda_1} = \frac{2\pi}{\sqrt{\alpha^2-1}}$, proving the lemma and the theorem. $\square$

3.4.11. Function as a section of trivial bundle (E, B, π)**.** This material is optional.

Before dealing with general bundles, let us make some elementary remarks about graphs of functions. Let M be a manifold. Consider a real-valued function $f\colon M \to \mathbb{R}$.

DEFINITION 3.4.38. The *graph* of f in $M \times \mathbb{R}$ is the collection of pairs

$$\{(p, y) \in M \times \mathbb{R} \colon \; y = f(p)\}.$$

We view $M \times \mathbb{R}$ as a trivial bundle (i.e., product bundle) over M, denoted π_M:

$$\pi_M \colon M \times \mathbb{R} \to M.$$

We can also consider the section $s\colon M \to M \times \mathbb{R}$ defined by the formula

$$s(p) = (p, f(p)).$$

Then the graph of f is the image of s in $M \times \mathbb{R}$. The section thus defined clearly satisfies the identity

$$\pi_M \circ s = Id_M,$$

in other words $\pi_M(s(p)) = p$ for all $p \in M$. We will now develop the language of bundles and sections in the context of the tangent and cotangent bundles of M.

The total space of a bundle is typically denoted E, the base is denoted B, and the surjective bundle projection is denoted π:

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ B \end{tikzcd}$$

The whole package is referred to as the triple

$$(E, B, \pi). \tag{3.4.20}$$

3.4.12. Möbius strip as a first example of a nontrivial bundle. This material is optional.

The tangent bundle of a circle is a cylinder $S^1 \times \mathbb{R}$ (see Proposition 2.2.6). The cylinder can be thought of as a *trivial* bundle over the circle. A first example of a *non-trivial* bundle is provided by the Möbius strip.[14] The latter can be thought of as a non-trivial bundle over the circle, constructed as follows. Consider the circle $C \subseteq \mathbb{R}^3$ parametrized by

$$\alpha(\theta) = 10(\cos\theta, \sin\theta, 0), \tag{3.4.21}$$

and its unit normal vector

$$n(\theta) = (\cos\theta, \sin\theta, 0).$$

The circle can be extended to a parametrized Möbius strip $M \subseteq \mathbb{R}^3$ as follows. We first construct the boundary of M by tracing the endpoints of a "rotating" interval at every point of the circle:

$$\mu(\theta) = \alpha(\theta) \pm \left(\cos\tfrac{\theta}{2}\, n(\theta) + \sin\tfrac{\theta}{2}\, e_3\right). \tag{3.4.22}$$

The resulting set (the image of μ) is the boundary of the Möbius strip. We now "fill in" the pair of points

$$\left(\cos\tfrac{\theta}{2}\, n(\theta) + \sin\tfrac{\theta}{2}\, e_3\right) \quad \text{and} \quad -\left(\cos\tfrac{\theta}{2}\, n(\theta) + \sin\tfrac{\theta}{2}\, e_3\right)$$

by the interval joining them:

$$s\left((\cos\tfrac{\theta}{2})\, n(\theta) + (\sin\tfrac{\theta}{2})\, e_3\right), \quad \text{where} \quad s \in [-1, 1]. \tag{3.4.23}$$

Remark 3.4.39. The interval is rotating at half the speed of the parametrisation of the circle (3.4.21). The result is that by the time we complete a full rotation around the circle, the interval will only be rotated by π.

We thus obtain a Möbius band with parametrisation $\underline{x}(\theta, s)$ defined as follows:

$$\underline{x}(\theta, s) = \alpha(\theta) + s\left(\left(\cos\tfrac{\theta}{2}\right)\, n(\theta) + \left(\sin\tfrac{\theta}{2}\right)\, e_3\right), \tag{3.4.24}$$

where $0 \le \theta \le 2\pi$ and $-1 \le s \le 1$.

Definition 3.4.40. The projection

$$\pi_M \colon M \to C \tag{3.4.25}$$

of M to the circle C collapses each interval

$$\alpha(\theta) + s\left(\left(\cos\tfrac{\theta}{2}\right)\, n(\theta) + \left(\sin\tfrac{\theta}{2}\right)\, e_3\right)$$

(for a fixed θ) to its midpoint $\underline{x}(\theta, 0) = \alpha(\theta) \in C$ defined by $s = 0$.

Theorem 3.4.41. *The map π_M of* (3.4.25) *defines a non-trivial interval bundle over the circle.*

[14] tabaat mobius, retzuat mobius.

PROOF. Consider the Möbius band parametrized as in (3.4.24). Its boundary ∂M is connected. Meanwhile the boundary of the cylinder $S^1 \times I$ has two connected components, each homeomorphic to a circle. Therefore the bundle π_M of (3.4.25) is not equivalent to the trivial bundle. □

3.4.13. Hairy ball theorem. This material is optional.

In Section 3.4.12, we discussed the simplest example of a nontrivial bundle, given by the Möbius band. Another example of a nontrivial bundle is provided by the following famous result.

THEOREM 3.4.42 (The hairy ball theorem). *There is no nonvanishing continuous tangent vector field on a sphere S^2.*

A proof can be obtained from Corollary 14.8.3.

Thus, if f is a continuous function on S^2 that assigns a vector in $\mathbb{R}^3$ to every point p on a sphere $S^2 \subseteq \mathbb{R}^3$, such that $f(p)$ is always tangent to the sphere at p, then there is at least one p such that $f(p) = 0$. The theorem was first stated by Henri Poincaré in the late 19th century.

The theorem is famously stated as "you can't comb a hairy ball flat", or sometimes, "you can't comb the hair of a coconut". It was first proved in 1912 by Brouwer. The theorem has the following corollary.

COROLLARY 3.4.43. *The tangent bundle of the sphere S^2 is a nontrivial bundle.*

PROOF. If the bundle were equivalent to the trivial bundle with fiber $\mathbb{R}^2$, then the constant vector field e_1 (the first basis vector of $\mathbb{R}^2$) would map to an everywhere nonvanishing vector field on S^2, contradicting the hairy ball theorem 3.4.42. □

CHAPTER 4

Differential forms, exterior derivative and algebra

To express the de Rham theorem and related results for a differentiable manifold M, we need to develop the language of differential k-forms on M.

4.1. Differential 1-form as section of cotangent bundle

In Section 2.2, we defined the tangent bundle of M. We defined the dual object, the cotangent bundle, in Section 2.7 as follows.

Definition 4.1.1. The cotangent bundle of M is the collection of pairs

$$T^*M = \left\{(p,\omega)\colon p \in M,\ \omega \in T_p^*\right\}$$

where T_p^* is the vector space dual to T_p.

Theorem 4.1.2. *The cotangent bundle of M^n is a $2n$-dimensional manifold.*

Proof. We have a natural projection $\pi_M\colon T^*M \to M$. Consider a coordinate chart $(u^1,\ldots,u^n)$ in a neighborhood $A \subseteq M$. We then obtain a basis, denoted $(du^1,\ldots,du^n)$, for the cotangent space at every point in A. Every cotangent vector ω_p at a point p with coordinates $(u^1,\ldots,u^n)$ is a linear combination

$$\omega_p = \omega_i du^i = \omega_1 du^1 + \omega_2 du^2 + \cdots + \omega_n du^n.$$

Hence we can coordinatize T^*M locally by the $2n$-tuple

$$(u^1,\ldots,u^n,\omega_1,\ldots,\omega_n),$$

proving the theorem. $\square$

With respect to this coordinate chart in T^*M, the standard projection $\pi_M\colon T^*M \to M$ "forgets" the last n coordinates:

$$\pi_M(u^1,\ldots,u^n,\omega_1,\ldots,\omega_n) = (u^1,\ldots,u^n).$$

Definition 4.1.3. A *differential* 1-*form* ω on M is a section of the cotangent bundle of M depending smoothly on $p \in M$, so that $\pi_M(\omega) = \mathrm{Id}_M$.

Thus at every point $p \in M$, a differential form ω gives a linear form $\omega_p \colon T_pM \to \mathbb{R}$. In coordinates $(u^1, \ldots, u^n)$, a differential 1-form ω is given by

$$\omega = \omega_i(u^1, \ldots, u^n)\, du^i,$$

where each of the ω_i is a real-valued function of n variables.

4.2. From function to differential 1-form

Definition 4.2.1. Given a differentiable manifold M, we denote by $C^\infty(M)$ the space of infinitely differentiable real-valued functions on M.

In Section 1.9 we defined the notion of a derivation on M, and the ring $\mathbb{D}_p$ of smooth functions defined near $p \in M$.

Definition 4.2.2. Let M be a differentiable manifold, $p \in M$. Given a smooth function $f \in C^\infty(M)$, we define a differential 1-form, denoted df, on M, as follows. For each vector field X we set

$$df(X) = Xf \tag{4.2.1}$$

at every $p \in M$ and $X_p \in T_pM$, where Xf denotes the evaluation of the derivation $X_p \in T_p$ at the function f thought of as an element of the ring $\mathbb{D}_p$.

Remark 4.2.3 (Relation to the gradient). The differential 1-form df is related to the gradient of f; the precise relation to the gradient will be clarified in Corollary 4.4.8 below.

Formula (4.2.1) applies at every point of M. Denoting $\omega = df$, we see that

$$\omega_p(X_p) = X_p f \in \mathbb{R}$$

where X_p is the value of the vector field X at the point p.

Theorem 4.2.4. *In coordinates $(u^1, \ldots, u^n)$ in a neighborhood $A \subseteq M$, we have*

$$df = \frac{\partial f}{\partial u^i} du^i, \tag{4.2.2}$$

with Einstein summation convention.

The theorem is a restatement of chain rule in several variables. The usual Leibniz rule for derivatives then implies the following.

Corollary 4.2.5 (Leibniz rule in terms of differential forms). *We have the following version of the Leibniz rule for functions on M:*

$$\forall f, g \in C^\infty(M), \quad d(fg) = f\, dg + g\, df. \tag{4.2.3}$$

Example 4.2.6. Let $u^1 = x$ and $u^2 = y$ for simplicity. Let $f(x, y) = x^2 + y^2$. Then the 1-form df is $df = 2x\, dx + 2y\, dy$.

4.3. Space $\Omega^1(M)$ of differential 1-forms, exterior derivative

Let ω be a 1-form on M, i.e., smooth map from M to T^*M satisfying $\pi_M \circ \omega = \mathrm{Id}_M$.

DEFINITION 4.3.1. We denote by

$$\Omega^1(M) = \{\omega \colon \pi_M \circ \omega = Id_M\}$$

the set of all differential 1-forms ω on M.

Note that $\Omega^1(M)$ is an infinite-dimensional vector space. The space $\Omega^1(M)$ is by definition the space of sections of the cotangent bundle of M. The 1-form df was defined in (4.2.1).

DEFINITION 4.3.2. The *exterior derivative* d on functions is the $\mathbb{R}$-linear map

$$d \colon C^\infty(M) \to \Omega^1(M), \quad f \mapsto df$$

from the space $C^\infty(M)$ of smooth functions on M to the space $\Omega^1(M)$ of differential 1-forms on M.

COROLLARY 4.3.3. *The exterior derivative d satisfies the Leibniz rule* (4.2.3).

4.4. Gradient & exterior derivative; musical isomorphisms

The exterior derivative is defined by the differentiable structure on the manifold M without recourse to metrics. Nevertheless it is easier to grasp the geometric significance of the exterior derivatives on functions by comparing df to the gradient of f, which does depend on the metric.

DEFINITION 4.4.1. Let $\big(V, \langle\ ,\ \rangle\big)$ be an inner product space. Let f be a function on V. The gradient ∇f of f is defined by setting

$$\langle \nabla f, X\rangle = Xf, \tag{4.4.1}$$

for every vector $X \in V$.

REMARK 4.4.2. This definition depends only on the metric and is independent of the coordinates chosen in V, since no coordinates were used in the definition (4.4.1). In other words, the gradient depends only on the inner product in V but not on the coordinates. With respect to an orthonormal basis the gradient of f is given by the list of partial derivatives of f.

DEFINITION 4.4.3 (Musical isomorphism bemol). Let $\big(V, \langle\ ,\ \rangle\big)$ be an inner product space. We define a map (bemol or flat)

$$\flat \colon V \to V^*, \quad X \mapsto X^\flat$$

by setting

$$\forall Y \in V, \quad X^\flat(Y) = \langle X, Y\rangle.$$

THEOREM 4.4.4. *Choose coordinates (u^i) in a neighborhood of the origin in the inner product space V in such a way that the standard basis $(\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n})$ is an orthonormal basis at the origin. Then*

$$\left(\frac{\partial}{\partial u^i}\right)^\flat = du^i.$$

In particular, the map $\flat$ is an isomorphism.

PROOF. Let $X = \frac{\partial}{\partial u^i}$. We write Y in coordinates as $Y = y^j \frac{\partial}{\partial u^j}$. Then

$$\langle X, Y\rangle = \left\langle \frac{\partial}{\partial u^i}, y^j \frac{\partial}{\partial u^j}\right\rangle = y^j \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right\rangle = y^j \delta_{ij} = y^i. \qquad (4.4.2)$$

Thus X extracts the i^{th} coordinate of Y via the inner product as in the formula (4.4.2), just as du^i does. Hence $X^\flat = du^i$. Since the 1-forms du^i, $i = 1, \ldots, n$ form a basis for V^* by Theorem 2.1.1, the result follows. □

DEFINITION 4.4.5 (Musical isomorphism diez). Let $\big(V, \langle\ ,\ \rangle\big)$ be an inner product space. The inverse of the isomorphism $\flat$ of Definition 4.4.3 is denoted

$$\sharp\colon V^* \to V, \quad \omega \mapsto \omega^\sharp,$$

(diez or sharp) for every 1-form ω.

COROLLARY 4.4.6. *In an inner product space, if the partial derivatives $\frac{\partial}{\partial u^i}$ form an orthonormal basis then*

$$\forall i, \quad (du^i)^\sharp = \frac{\partial}{\partial u^i}.$$

Let M be a Riemannian manifold, i.e., a differentiable manifold equipped with a metric (first fundamental form), i.e., a bilinear form at each point of M; see Section 3.1.

DEFINITION 4.4.7 (Musical isomorphisms on manifold). We use the first fundamental form on T_p to define isomorphisms $\flat\colon T_p \to T_p^*$ and $\sharp\colon T_p^* \to T_p$ at every point $p \in M$.

Then Corollary 4.4.6 can be formulated as follows. Recall that relative to an orthonormal basis $\big(\frac{\partial}{\partial u^i}\big)$, the gradient ∇f of f is given by the n-tuple $(\frac{\partial f}{\partial u^i})$ where $i = 1, \ldots, n$.

COROLLARY 4.4.8. *Let M be a Riemannian manifold. The exterior derivative of a function f at $p \in M$ is related to the gradient ∇f of f at $p \in M$ as follows:*

$$\nabla f = \sharp(df), \quad df = \flat(\nabla f). \tag{4.4.3}$$

at every point p of the manifold M, where the musical isomorphisms are determined by the metric.

PROOF. We choose coordinates (u^i) in a neighborhood of p in such a way that at the point p itself, the basis $(\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n})$ is orthonormal. Then the result follows from linearity by checking for each element of the basis. □

REMARK 4.4.9. Formula (4.4.3) is basis-independent (even though the proof is carried out in an orthonormal basis), since both the gradient of f and df are defined in a way independent of the basis; see Remark 4.4.2.

We will now develop material in linear algebra that constitutes necessary preliminaries for the definition of de Rham cohomology.[1]

4.5. Exterior product and algebra, alternating property

The exterior algebra is sometimes referred to as the Grassmann algebra.[2]

[1] Plan for building de Rham cohomology. This material is optional. The construction of de Rham cohomology of a differentiable manifold M involves several stages. It may be helpful to keep these stages in mind as we build up the relevant mathematical machinery step-by-step. We start with linear algebra and end with linear algebra.

(1) Linear-algebraic stage: from a vector space V to its exterior algebra $\bigwedge V = \oplus_k \bigwedge^k V$.
(2) Topological stage: from a smooth manifold M to its tangent bundle TM and its cotangent bundle T^*M.
(3) A vector field as a section of TM.
(4) A section of T^*M is a differential 1-form on M.
(5) The exterior bundle $\bigwedge M = \oplus_k \bigwedge^k (T^*M)$ is the bundle of exterior algebras parametrized by points of M. This is a finite-dimensional manifold.
(6) A section of $\bigwedge^k (T^*M)$ is a differential k-form on M.
(7) The space $\Omega^k(M)$ of sections of $\bigwedge^k (T^*M)$ is an infinite-dimensional vector space: we are back to linear algebra.
(8) The exterior derivative d turns $\Omega^*(M)$ into a differential graded algebra.
(9) De Rham cohomology groups are defined in terms of the differential graded algebra $(\Omega^*(M), d)$.

[2] See `https://mathoverflow.net/questions/22247/geometrical-meaning-of-grassmann-algebra` for a motivating discussion.

Generalizing the notion of differential 1-form to differential k-forms on a differentiable manifold M requires certain linear-algebraic preliminaries concerning the exterior algebra. Some preliminary remarks appear in the note.[3]

DEFINITION 4.5.1. Let V be a vector space over a field K. The exterior algebra over V is a unital associative[4] algebra over the field K, that includes V itself as a subspace. Such an algebra is denoted by

$$\left(\bigwedge(V), \wedge\right),$$

where $\wedge$ is the wedge product operation in the algebra. Thus we have $V \subseteq \bigwedge(V)$.

REMARK 4.5.2 (Construction). We will begin defining the exterior algebra in Section 4.6 and provide examples. A construction of the exterior algebra appears in Definition 5.4.1.

DEFINITION 4.5.3. The *wedge product* is an *associative* and *bilinear* operation:

$$\wedge\colon \bigwedge(V) \times \bigwedge(V) \to \bigwedge(V), \quad (\alpha, \beta) \mapsto \alpha \wedge \beta,$$

with the essential feature that it is *anticommutative* for elements of V, meaning that

$$\forall v \in V, \quad v \wedge v = 0. \tag{4.5.1}$$

REMARK 4.5.4. Property (4.5.1) implies in particular

$$u \wedge v = -v \wedge u \tag{4.5.2}$$

for all $u, v \in V$, and

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k = 0 \tag{4.5.3}$$

whenever $v_1, \ldots, v_k \in V$ are linearly dependent.

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[3]The following remarks provide some motivation for the sequel.

(1) The exterior product, or wedge product, of vectors is an algebraic construction generalizing certain features of the *cross product* to higher dimensions.
(2) (Basis-independent) In linear algebra, the exterior product provides a basis-independent manner for describing the *determinant* and the *minors* of a linear transformation.
(3) The exterior algebra over a vector space V is generated by an operation called *exterior product*. It is a key ingredient in the definition of the algebra of differential forms.
(4) (Sign change) Just as the determinant of a matrix changes sign when we switch a pair rows, the wedge product is anti-commutative in the sense specified below.

[4]chok kibutz

We note the following three points:

(1) associativity is required for all elements of the algebra $\bigwedge(V)$;
(2) bilinearity is required for all elements of the algebra $\bigwedge(V)$;
(3) the three properties (4.5.1), (4.5.2), (4.5.3) are satisfied only on the elements of the subspace $V \subseteq \bigwedge(V)$.

The defining property (4.5.1) and property (4.5.3) are equivalent; properties (4.5.1) and (4.5.2) are equivalent unless the characteristic of K is two.

4.6. Exterior algebra over a dim 1 vector space

We assume the existence of such an algebra and derive some of its properties. A general construction of the algebra will be provided in Section 5.4.

THEOREM 4.6.1. *Let V be a 1-dimensional vector space over $\mathbb{R}$. Let $\bigwedge(V)$ be the exterior algebra over V. Then*

(1) *the algebra is 2-dimensional;*
(2) *the algebra is isomorphic to the subalgebra of the algebra of 2×2 matrices consisting of uppertriangular*[5] *matrices with a pair of identical eigenvalues:*

$$\bigwedge(V) \simeq \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

PROOF. Let Id denote the identity matrix. Let n denote the matrix

$$n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

As a vector space, the algebra is the direct sum

$$\bigwedge(V) \simeq \mathbb{R} \cdot Id + \mathbb{R} \cdot n.$$

The matrix n is nilpotent: $n^2 = 0$, as required by (4.5.1). Here the vector space V is identified with the line $\mathbb{R}\, n \subseteq \bigwedge(V)$. □

DEFINITION 4.6.2. One uses the notation

$$\begin{cases} \bigwedge^0(V) = \mathbb{R}\, Id \\ \bigwedge^1(V) = V = \mathbb{R}\, n, \end{cases}$$

so that $\bigwedge(V) = \bigwedge^0(V) + \bigwedge^1(V)$.

[5] meshulashit

4.7. Areas in the plane; signed area

The purpose of this section is to motivate the skewness of the exterior product $\wedge$ on vectors in $V \subseteq \bigwedge(V)$ based on geometric considerations of areas.

EXAMPLE 4.7.1. The parallelogram spanned by vectors $v, w \in \mathbb{R}^2$ has area equal to the absolute value of the determinant of the matrix $\begin{pmatrix} v^1 & w^1 \\ v^2 & w^2 \end{pmatrix}$ of the coordinates of the vectors.

In more detail, the Cartesian plane $\mathbb{R}^2$ is a vector space equipped with a basis. The basis consists of a pair of unit vectors $\mathbf{e}_1 = (1,0)^t$ and $\mathbf{e}_2 = (0,1)^t$. Suppose that

$$v = v^1\mathbf{e}_1 + v^2\mathbf{e}_2, \quad w = w^1\mathbf{e}_1 + w^2\mathbf{e}_2$$

are a pair of vectors in $\mathbb{R}^2$, written in components. There is a unique parallelogram having v and w as two of its sides. The area of this parallelogram is given by the standard determinant formula:

$$\begin{aligned} A &= \left|\det \begin{bmatrix} v & w \end{bmatrix}\right| \\ &= |v^1w^2 - v^2w^1|. \end{aligned}$$

Consider now the exterior product $\wedge$ of v and w and exploit the properties stipulated above:

$$\begin{aligned} v \wedge w &= (v^1\mathbf{e}_1 + v^2\mathbf{e}_2) \wedge (w^1\mathbf{e}_1 + w^2\mathbf{e}_2) \\ &= v^1w^1\mathbf{e}_1 \wedge \mathbf{e}_1 + v^1w^2\mathbf{e}_1 \wedge \mathbf{e}_2 + v^2w^1\mathbf{e}_2 \wedge \mathbf{e}_1 + v^2w^2\mathbf{e}_2 \wedge \mathbf{e}_2 \\ &= (v^1w^2 - v^2w^1)\mathbf{e}_1 \wedge \mathbf{e}_2, \end{aligned}$$

where the first step uses the distributive law for the wedge product, and the last uses the fact that the wedge product is alternating. Thus,

$$v \wedge w = (v^1w^2 - v^2w^1)\mathbf{e}_1 \wedge \mathbf{e}_2. \tag{4.7.1}$$

REMARK 4.7.2 (Signed area). The coefficient in (4.7.1) is precisely the determinant of the matrix $[v\ w]$. The fact that this may be positive or negative has the intuitive meaning that v and w may be oriented in a counterclockwise or clockwise sense as the vertices of the parallelogram they define.

The coefficient is called the *signed area* of the parallelogram: the absolute value of the signed area is the ordinary area, and the sign determines its orientation.

4.7.1. Algebraic characterisation of signed area. The material in this subsection and the one following is optional.

If $A(v, w)$ denotes the signed area of the parallelogram spanned by the pair of vectors v and w, then A must have the following properties. This axiomatization of areas is due to Leopold Kronecker and Karl Weierstrass.

(1) $A(av, bw) = abA(v, w)$ for any real numbers a and b, since rescaling either of the sides rescales the area by the same amount (and reversing the direction of one of the sides reverses the orientation of the parallelogram).
(2) $A(v, v) = 0$, since the area of the degenerate parallelogram determined by v, v (i.e., a line segment) is zero.
(3) $A(w, v) = -A(v, w)$, since interchanging the roles of v and w reverses the orientation of the parallelogram.
(4) $A(v + aw, w) = A(v, w)$, since adding a multiple of w to v affects neither the base nor the height of the parallelogram and consequently preserves its area.
(5) $A(e_1, e_2) = 1$, since the area of the unit square is one.

With the exception of the last property, the wedge product satisfies the same formal properties as the signed area. In a certain sense, the wedge product generalizes the final property by allowing the area of a parallelogram to be compared to that of any "standard" chosen parallelogram.

REMARK 4.7.3. The exterior product in two-dimensions is a basis-independent formulation of area.

4.7.2. Vector product, triple product, and wedge product. For a 3-dimensional vector space V over $\mathbb{R}$, the wedge product in the exterior algebra $\bigwedge(V)$ is closely related to the vector product and triple product.[6]

EXAMPLE 4.7.4. Using the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$, the wedge product of a pair of vectors $\mathbf{u} = u^1\mathbf{e}_1 + u^2\mathbf{e}_2 + u^3\mathbf{e}_3$ and $\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3$ is $\mathbf{u}\wedge\mathbf{v} = (u^1v^2 - u^2v^1)(\mathbf{e}_1\wedge\mathbf{e}_2) + (u^1v^3 - u^3v^1)(\mathbf{e}_1\wedge\mathbf{e}_3) + (u^2v^3 - u^3v^2)(\mathbf{e}_2\wedge\mathbf{e}_3)$ (don't try to change the signs here), where $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ is the basis for the three-dimensional space $\bigwedge^2(\mathbb{R}^3)$ (in notation similar to that of Definition 4.6.2). This formula is similar to the usual definition of the vector product of vectors in three dimensions.

EXAMPLE 4.7.5. Consider a third vector $\mathbf{w} = w^1\mathbf{e}_1 + w^2\mathbf{e}_2 + w^3\mathbf{e}_3$. Then the wedge product of three vectors is

$$\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w} = (u^1v^2w^3 + u^2v^3w^1 + u^3v^1w^2 - u^1v^3w^2 - u^2v^1w^3 - u^3v^2w^1)(\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3),$$

where $e_1 \wedge e_2 \wedge e_3$ is the basis vector for the one-dimensional space $\bigwedge^3(\mathbb{R}^3)$. Note that the coefficient is the usual triple product $(u \times v) \cdot w$.

The vector product and triple product in three dimensions each admit both geometric and algebraic interpretations.

[6] machpela meurevet

(1) (geometric interpretation) The vector product $u \times v$ can be interpreted as a vector which is perpendicular to both u and v and whose magnitude is equal to the *area* of the parallelogram determined by the two vectors.
(2) (algebraic interpretation) The vector product can also be interpreted as the vector consisting of the minors of the matrix with columns u and v.

REMARK 4.7.6. The triple product of u, v, and w is geometrically a (signed) volume. It is also the determinant of the matrix with columns u, v, and w.

The exterior product in three dimensions allows for similar interpretations. In fact, in the presence of a positively oriented orthonormal basis, the exterior product generalizes these notions to higher dimensions.

4.8. Anticommutativity of the wedge product

THEOREM 4.8.1. *Assume $v \wedge v = 0$ for all $v \in V$. Then the wedge product is anticommutative on elements of $V \subseteq \bigwedge(V)$ in the sense that $u \wedge v = -v \wedge u$ for all $u, v \in V$.*

PROOF. Let $x, y \in V$. Then $0 = (x + y) \wedge (x + y) = x \wedge x + x \wedge y + y \wedge x + y \wedge y = x \wedge y + y \wedge x$. Hence $x \wedge y = -y \wedge x$. □

COROLLARY 4.8.2. *If $x_1, x_2, ..., x_k$ are elements of V, and σ is a permutation of the integers $(1, 2, \ldots, k)$, then*

$$x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \cdots \wedge x_{\sigma(k)} = \operatorname{sgn}(\sigma) x_1 \wedge x_2 \wedge \cdots \wedge x_k,$$

where $sgn(\sigma)$ is the sign (plus or minus) of a permutation $\sigma \in S_k$.

A proof can be found in greater generality in Bourbaki (1989).

4.9. The k-exterior power; simple multivectors

DEFINITION 4.9.1. The k-th exterior power of V, denoted $\bigwedge^k(V)$, is the vector subspace of $\bigwedge(V)$ spanned by elements of the form

$$x_1 \wedge x_2 \wedge \cdots \wedge x_k, \quad x_i \in V, \quad i = 1, 2, \ldots, k.$$

DEFINITION 4.9.2. An element $\alpha \in \bigwedge^k(V)$ is said to be a k-multivector.

DEFINITION 4.9.3. If α can be expressed as a wedge product of k elements of V, then α is said to be *decomposable*, or *simple*.

Although decomposable multivectors span $\bigwedge^k(V)$, not every element of $\bigwedge^k(V)$ is decomposable.

EXAMPLE 4.9.4. In $\mathbb{R}^4$, the following 2-multivector is not decomposable:

$$\alpha = e_1 \wedge e_2 + e_3 \wedge e_4. \tag{4.9.1}$$

In the sequel, this α will be referred to as the symplectic form, possessing the property $\alpha \wedge \alpha \neq 0$.

4.10. Basis and dimension of exterior algebra

THEOREM 4.10.1. *If the dimension of V is n and $e_1, ..., e_n$ is a basis of V, then the set*

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \colon 1 \leq i_1 < i_2 < \cdots < i_k \leq n\} \tag{4.10.1}$$

is a basis for $\bigwedge^k(V)$.

PROOF. Given any wedge product of the form $v_1 \wedge \cdots \wedge v_k$, every vector v_j can be written as a linear combination of the basis vectors e_i. Using the bilinearity of the wedge product, the expression $v_1 \wedge \cdots \wedge v_k$ can be expanded to a linear combination of wedge products of such basis vectors. Any wedge product in which the same basis vector appears more than once is zero. Any wedge product in which the basis vectors do not appear in the proper order can be reordered, changing the sign whenever two basis vectors change places. $\square$

By counting the basis elements, we obtain the following corollary.

COROLLARY 4.10.2. *The dimension of $\bigwedge^k(V)$ is the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. In particular, $\bigwedge^k(V) = 0$ for $k > n$.*

THEOREM 4.10.3. *The dimension of $\bigwedge(V)$ equals 2^n.*

PROOF. Any element of the exterior algebra can be written as a sum of multivectors. Hence, as a vector space the exterior algebra is a direct sum

$$\bigwedge(V) = \bigwedge^0(V) + \bigwedge^1(V) + \bigwedge^2(V) + \cdots + \bigwedge^n(V)$$

(where by convention $\bigwedge^0(V) = \mathbb{R}$ and $\bigwedge^1(V) = V$), and therefore its dimension is equal to the sum of the binomial coefficients (as k runs from 1 to n), which is 2^n. $\square$

CHAPTER 5

Exterior differential complex

We define the exterior differential complex of a manifold. Our eventual goal is to build the de Rham cohomology of M in Section 7.9.

5.1. Rank of a multivector

In Section 4.9 we defined the k-th exterior power $\bigwedge^k(V)$ of a vector space V. Let $\alpha \in \bigwedge^k(V)$ be a k-multivector. Thus, α is a linear combination of a finite number, say s, of decomposable (simple) multivectors:

$$\alpha = \alpha^{(1)} + \alpha^{(2)} + \cdots + \alpha^{(s)} \tag{5.1.1}$$

meaning that each $\alpha^{(i)}$ is decomposable into a wedge product of the following form:

$$\alpha^{(i)} = \alpha_1^{(i)} \wedge \cdots \wedge \alpha_k^{(i)}, \quad i = 1, 2, \ldots, s$$

where $\alpha_j^{(i)} \in \bigwedge^1(V) = V$ for all $i = 1, \ldots, s$ and $j = 1, \ldots, k$.

Definition 5.1.1. The *rank* $\operatorname{rank}(\alpha)$ of the multivector α is the minimal number s of decomposable multivectors in all possible expansions (5.1.1) of α.

5.2. Rank of 2-multivectors; matrix of coefficients

Let $k = 2$. Let $(e_i)_{i=1,\ldots,n}$ be a basis for V. Let $\alpha \in \bigwedge^2(V)$ be a multivector. Thus α can be expressed uniquely as

$$\alpha = \sum_{i<j} a_{ij}\, e_i \wedge e_j \tag{5.2.1}$$

where we use only the pairs of indices with $i < j$.

Definition 5.2.1. Let α be a 2-multivector as in (5.2.1). The *matrix of coefficients* of the 2-multivector α is the antisymmetric matrix $A_\alpha = (a_{ij})$ where by definition $a_{ji} = -a_{ij}$ whenever $i < j$.

For further details see Example 6.7.1.

THEOREM 5.2.2. *The rank of a 2-multivector α equals* half *the rank of its matrix of coefficients A_α:*

$$\operatorname{rank}(\alpha) = \frac{1}{2}\operatorname{rank}(A_\alpha).$$

PROOF. This follows from the fact that every antisymmetric real matrix can be orthogonally diagonalized into 2 by 2 blocks; see Theorem 6.9.1. □

EXAMPLE 5.2.3. The symplectic form $\alpha = e_1 \wedge e_2 + e_3 \wedge e_4 \in \bigwedge^2 \mathbb{R}^4$ of (4.9.1) has rank 2, whereas its matrix of coefficients

$$A_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

is of rank 4. Thus $\operatorname{rank}\alpha = \frac{1}{2}\operatorname{rank} A_\alpha$.

For further details see Example 6.7.1. We mention the following theorem for general culture.

THEOREM 5.2.4. *Over a field of characteristic 0, a 2-multivector α has rank p if and only if the p-fold product satisfies*

$$\underbrace{\alpha \wedge \cdots \wedge \alpha}_{p} \neq 0$$

and

$$\underbrace{\alpha \wedge \cdots \wedge \alpha}_{p+1} = 0.$$

5.3. Construction of the tensor algebra

In Chapter 4, the exterior algebra was introduced *axiomatically*, i.e., characterized via its properties. In Section 5.4, we will provide a *construction* of the exterior algebra via tensor products.

The tensor product $V \otimes W$ of two vector spaces V and W over the field $\mathbb{R}$ is defined as follows. Consider the set of ordered pairs (v, w) in the Cartesian product $V \times W$.

REMARK 5.3.1. For the purposes of the construction, we regard the Cartesian product as its underlying *set* rather than a vector space.

DEFINITION 5.3.2. A typical element of $V \times W$ viewed as a set will be denoted $e_{(v,w)}$ for the purposes of the construction that follows.

DEFINITION 5.3.3. The *free vector space* $F = F(V \times W)$ on $V \times W$ is the vector space in which the elements $e_{(v,w)}$ of $V \times W$ are a basis.

Thus, $F(V \times W)$ is the collection of all finite linear combinations and can be written as follows:

$$F(V \times W) = \left\{ \sum_i \alpha_i e_{(v_i, w_i)} \colon \alpha_i \in \mathbb{R}, v_i \in V, w_i \in W \right\}.$$

The terms $e_{(v,w)}$ are by definition linearly independent in $F(V \times W)$ for distinct pairs $(v, w) \in V \times W$. The tensor product arises by enforcing the following equivalences on the free vector space $F(V \times W)$:

$$\begin{aligned} e_{(v_1+v_2,w)} &\sim e_{(v_1,w)} + e_{(v_2,w)} \\ e_{(v,w_1+w_2)} &\sim e_{(v,w_1)} + e_{(v,w_2)} \\ ce_{(v,w)} &\sim e_{(cv,w)} \\ ce_{(v,w)} &\sim e_{(v,cw)} \end{aligned} \tag{5.3.1}$$

where $v_1, v_2 \in V$, $w_1, w_2 \in W$, and $c \in K$.

Definition 5.3.4. Let $S \subseteq F(V \times W)$ be the vector subspace generated by the four equivalence relations (5.3.1); in other words, by all the differences

$e_{(v_1+v_2,w)} - \big(e_{(v_1,w)} + e_{(v_2,w)}\big)$,
$e_{(v,w_1+w_2)} - \big(e_{(v,w_1)} + e_{(v,w_2)}\big)$,
$ce_{(v,w)} - e_{(cv,w)}$,
$ce_{(v,w)} - e_{(v,cw)}$.

The equivalence relation $\sim$ among elements $\alpha, \beta \in F(V \times W)$ is defined by

$$\alpha \sim \beta \quad \text{if and only if} \quad \alpha - \beta \in S. \tag{5.3.2}$$

Definition 5.3.5. The expression $\left[\sum_i \alpha_i e_{(v_i,w_i)}\right]$ denotes the equivalence class of the finite sum $\sum_i \alpha_i e_{(v_i,w_i)}$ relative to the equivalence relation (5.3.2).

Definition 5.3.6. The tensor product of vector spaces V and W can be described in the following two equivalent ways:

(1) the quotient space $V \otimes W = F(V \times W)/S$;
(2) $V \otimes W = \left\{ \left[\sum_i \alpha_i e_{(v_i,w_i)}\right] : \alpha_i \in \mathbb{R}, v_i \in V, w_i \in W \right\}$.

Definition 5.3.7. The equivalence class of the element $e_{(v,w)}$ in $V \otimes W$ will be denoted $v \otimes w$.

Theorem 5.3.8. *The dimensions of V and W multiply under tensor product:*

$$\dim(V \otimes W) = \dim V \dim W.$$

Proof. Let $(e_1, \ldots, e_n)$ be a basis for V. Let $(f_1, \ldots, f_m)$ be a basis for W. Then the mn elements $e_i \otimes f_j$, where $i = 1, \ldots, n$, $j = 1, \ldots, m$ form a basis for the space $V \otimes W$. $\square$

Remark 5.3.9. While bases are useful in computing dimensions, the advantage of the construction via the free vector space lies in its independence of the choice of basis.

Definition 5.3.10. The product

$$V^{\otimes k} \times V^{\otimes \ell} \to V^{\otimes (k+\ell)}$$

is defined on basis vectors by sending the pair $(v_1 \otimes \cdots \otimes v_k,\ w_1 \otimes \cdots \otimes w_\ell)$ to $(v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell)$.

5.4. Construction of the exterior algebra

We will now define the exterior powers of a vector space V in terms of the tensor products of Section 5.3.

Definition 5.4.1. The exterior product $V \wedge V$ of V by itself is the vector space obtained as the quotient of $V \otimes V$ by the vector subspace generated by all the sums $v \otimes w + w \otimes v$ where $v, w \in V$.

Definition 5.4.2. The image of $v \otimes w$ under the surjective homomorphism $V \otimes V \to V \wedge V$ is denoted $v \wedge w$.

Given a basis (e_i) for V, the products $e_i \wedge e_j$, where $i < j$, form a basis for the space $V \wedge V = \bigwedge^2(V)$.

Definition 5.4.3. The third exterior power $\bigwedge^3 V$ is defined as the quotient of $V \otimes V \otimes V$ by the subspace generated by differences of the form

$$u_1 \otimes u_2 \otimes u_3 - \operatorname{sgn}(\sigma) u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes u_{\sigma(3)} \tag{5.4.1}$$

for all $\sigma \in S_3$.

The higher exterior powers $\bigwedge^k(V)$ are defined similarly as quotients of $V \otimes V \otimes \ldots \otimes V$ (k times) by imposing an anticommutation property generalizing (5.4.1).

Definition 5.4.4. The wedge product $\wedge \colon \bigwedge^k V \times \bigwedge^\ell V \to \bigwedge^{k+\ell} V$ is induced from the tensor product $\otimes^k V \times \otimes^\ell V \to \otimes^{k+\ell} V$ by quotienting as above (choose representing k-fold and ℓ-fold tensors, multiply them in the tensor algebra, and take the equivalence class in $\bigwedge^{k+\ell}$).

5.5. Exterior bundle, differential form, exterior derivative

Let M be a differentiable manifold M of dimension n.
Let $\pi_M \colon T^*M \to M$ be its contangent bundle.
Let $p \in M$ be a point.

DEFINITION 5.5.1. Associating to every cotangent space T_p^* its exterior algebra $\bigwedge(T_p^*)$ and taking disjoint union, we obtain the exterior bundle

$$\bigwedge M = \bigwedge(T^*M),$$

with a canonical projection $\pi_M \colon \bigwedge M \to M$ with typical fiber $\bigwedge(T_p^*)$ of dimension 2^n.

As for the tangent and cotangent bundles, one can exhibit local charts and transition functions to demonstrate local triviality of the exterior bundle.

DEFINITION 5.5.2. Putting together the k-exterior powers $\bigwedge^k T_p^*$, we obtain the subbundle $\bigwedge^k M = \bigwedge^k(T^*M)$ of the bundle $\bigwedge(M)$.

DEFINITION 5.5.3. A *differential k-form* on a manifold M is a section of the exterior bundle $\bigwedge^k M$ of k-multivectors built from elements of T^*M.

DEFINITION 5.5.4. Let $\Omega^k(M)$ be the space of differential k-forms on M.

Recall that we have an exterior derivative d defined on smooth functions $f \in C^\infty(M)$ locally in a coordinate chart $(A, (u^1, \ldots, u^n))$ by the formula

$$df = \frac{\partial f}{\partial u^i} du^i, \tag{5.5.1}$$

with Einstein summation convention; see (4.2.2). We next define the exterior derivative on 1-forms.

DEFINITION 5.5.5. The exterior derivative, or differential,

$$d \colon \Omega^1(M) \to \Omega^2(M)$$

is defined on a 1-form $f(u^1, \ldots, u^n)du$ by setting

$$d(f du) = df \wedge du. \tag{5.5.2}$$

REMARK 5.5.6. It is immediate from the definition that for each constant form du^i one has $d(du^i) = 0$.

REMARK 5.5.7 (Issue of signs). If one views the form $f du$ as the product $(du)f$, one would need to introduce a sign in order to be compatible with formula (5.5.2):

$$d\big((du)f\big) = -du \wedge df$$

by the basic property of 1-forms: $df \wedge du = -du \wedge df$.

A similar formula defines the exterior derivative for an arbitrary k-form.

DEFINITION 5.5.8. For a k-form $\omega = f du^{i_1} \wedge \cdots \wedge du^{i_k} \in \Omega^k(M)$, one sets

$$d(\omega) = df \wedge du^{i_1} \wedge \cdots \wedge du^{i_k} \in \Omega^{k+1}(M).$$

The general form of the Leibniz rule for differential forms appears in (5.7.1) in Section 5.7.

5.6. Pullback of differential forms

PROPOSITION 5.6.1. *Consider a smooth map $\phi\colon M \to N$ between differentiable manifolds. Then ϕ defines a natural map*

$$d\phi\colon TM \to TN$$

called the tangent map.

PROOF. As in Section 2.5, we represent a tangent vector $v \in T_pM$ by a path $c(t)\colon I \to M$, such that $c'(0) = v$. The composite map $\sigma = \phi \circ c\colon I \to N$ is a path in N. Then the vector $\sigma'(0)$ is the image of v under $d\phi$:

$$d\phi(v) = \sigma'(0),$$

proving the proposition. □

DEFINITION 5.6.2 (Pullback form). Let $\phi : M \to N$ be a smooth map between differentiable manifolds, and let $\omega \in \Omega^2(N)$ be a differential 2-form on the target N. The *pullback* form $\phi^*(\omega)$ on M is the 2-form defined by

$$\phi^*(\omega)(X, Y) = \omega(d\phi(X), d\phi(Y)),$$

where $d\phi$ is the tangent map of Proposition 5.6.1.

Pullback of differential k-forms is defined similarly. We therefore obtain the following theorem.

THEOREM 5.6.3 (Pullback homomorphism). *A differentiable map $f\colon M \to N$ induces a natural "pullback" homomorphism $f^*\colon \Omega^k(N) \to \Omega^k(M)$ for each k.*

5.7. Exterior differential complex

Differential k-forms were defined in Section 5.5. We will define the de Rham cohomology of a differentiable manifold M of dimension n by means of the exterior differential complex.

THEOREM 5.7.1. *The exterior derivative d gives rise to an exterior differential complex which is by definition the following sequence of homomorphisms:*

$$0 \to C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(M) \to 0,$$

with the following properties:

(1) *one has $d^2 = d \circ d = 0$ at each stage of the complex;*

(2) *one has the following form of the Leibniz superrule for d applied to differential forms:*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \, \alpha \wedge d\beta, \tag{5.7.1}$$

where $p = \deg(\alpha)$ is the degree of α.

COROLLARY 5.7.2. *If the form α has even degree then the sign disappears and we obtain the naive Leibniz rule $d(\alpha \wedge \beta) = d\alpha \wedge \beta + \alpha \wedge d\beta$, regardless of the degrees of α and β.*

PROOF OF THEOREM 5.7.1. Let us check the property $d^2 = 0$ listed in item (1) as applied to a typical 1-form $\omega = f du$. Thus,

$$d\omega = df \wedge du = \frac{\partial f}{\partial u^i}\, du^i \wedge du.$$

Then we exploit the equality of mixed second partial derivatives to write

$$\begin{aligned} d(d\omega) &= d\left(\frac{\partial f}{\partial u^i} du^i \wedge du\right) \\ &= \frac{\partial^2 f}{\partial u^i \partial u^j} du^j \wedge du^i \wedge du \\ &= \sum_{i<j} \frac{\partial^2 f}{\partial u^i \partial u^j} \left(du^j \wedge du^i \wedge du + du^i \wedge du^j \wedge du\right) \\ &= \sum_{i<j} \frac{\partial^2 f}{\partial u^i \partial u^j} \left(du^j \wedge du^i + du^i \wedge du^j\right) \wedge du \\ &= 0 \end{aligned}$$

by the basic anticommutation relation $du^j \wedge du^i + du^i \wedge du^j = 0$ among 1-forms.

To prove the Leibniz superrule (2), consider first 1-forms $\alpha = f du$ and $\beta = g\, dv$. Then

$$\begin{aligned} d(\alpha \wedge \beta) &= d(f du \wedge g\, dv) \\ &= d(fg\, du \wedge dv) \\ &= (dfg + fdg) \wedge du \wedge dv \\ &= dfg \wedge du \wedge dv + fdg \wedge du \wedge dv \\ &= df \wedge du \wedge (g\, dv) - (f du) \wedge dg \wedge dv) \\ &= d(f du) \wedge (g\, dv) - (f du) \wedge d(g\, dv) \\ &= d\alpha \wedge \beta - \alpha \wedge d\beta. \end{aligned}$$

The general case is treated similarly. □

Remark 5.7.3. The exterior differential complex leads to the definition of de Rham cohomology of M; see Section 7.7.

5.8. Antisymmetric multilinear functions

In this section we develop an equivalent definition of the k-exterior power of a vector space V, in terms of alternating (antisymmetric) multilinear functions. The equivalent approach is useful for explicit computations.

Definition 5.8.1. Let V be a vector space over $\mathbb{R}$. Let $k \in \mathbb{N}$. An antisymmetric multilinear function

$$f\colon V^k \to \mathbb{R}, \tag{5.8.1}$$

is a function satisfying $f(v_{\sigma(1)}, \ldots v_{\sigma(k)}) = \operatorname{sign}(\sigma) f(v_1, \ldots, v_k)$, for all $v_1, \ldots, v_k \in V$ and $\sigma \in S_k$.

Lemma 5.8.2. *The set of all antisymmetric k-multilinear functions is a vector space.*

Namely, the sum of two such maps and the product of such a map by a scalar are again antisymmetric.

Example 5.8.3. Let $k = 1$. Then the antisymmetric condition is vacuous and the space of antisymmetric functions is simply the dual space $V^* = \bigwedge^1(V^*)$.

In general, we have the following duality.

Theorem 5.8.4. *If V has finite dimension n, then the space of antisymmetric k-multilinear functions on V is naturally identified with the k-th exterior product $\bigwedge^k(V^*)$ of the dual space V^*.*

PROOF. Let us make such an identification explicit. Consider a k-tuple

$$x = (x_1, \ldots, x_k) \in V^k.$$

Elements of the dual space V^* will be denoted y^i. Consider a decomposable (simple) k-multivector

$$y = y^1 \wedge \ldots \wedge y^k \in \bigwedge^k V^*.$$

We would like to define an antisymmetric multilinear map $f = f_y$ as in (5.8.1), associated with the multivector y. We define such a map by setting

$$f_y(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, y^1(x_{\sigma(1)}) \ldots y^k(x_{\sigma(k)}). \tag{5.8.2}$$

In other words,

$$f_y(x) = \det \left(y^i(x_j) \right)_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}}$$

is a $k \times k$-determinant. The antisymmetric property follows from a similar property of the determinant under permutations of columns. Note that we do not multiply by $\frac{1}{k!}$. □

COROLLARY 5.8.5. *The dimension of the space of antisymmetric multilinear maps from V^k to $\mathbb{R}$ is the binomial coefficient $\binom{n}{k}$.*

Recall that V^* is the dual of V. We have a similar property for exterior powers.

THEOREM 5.8.6. *The space $\bigwedge^k(V^*)$ is naturally dual to $\bigwedge^k(V)$.*

PROOF. We view an element of $\bigwedge^k(V^*)$ as an antisymmetric k-multilinear function ϕ. The duality is given on simple (decomposable) elements $v_1 \wedge \ldots \wedge v_k$ of $\bigwedge^k(V)$ by the pairing

$$\langle \phi\, ,\, v_1 \wedge \ldots \wedge v_k \rangle = \phi(v_1, \ldots, v_k).$$

The pairing is extended by linearity to all of $\bigwedge^k V$. One readily checks the independence of the resulting value of the particular representation of the simple multivector as a product (see Proposition 5.9.2 for more details). □

The case $k = 2$ will be examined in detail in the next section.

5.9. Case of 2-forms

Let $y^1, y^2 \in V^*$ be 1-forms on V. Consider the decomposable (simple) exterior 2-form

$$\omega = y^1 \wedge y^2 \in \bigwedge^2 (V^*).$$

Then ω defines an antisymmetric bilinear map

$$f_\omega \colon V^2 \to \mathbb{R}$$

defined as follows. Let $u, v \in V$. Following (5.8.2), we set

$$f_\omega(u, v) = y^1(u)y^2(v) - y^1(v)y^2(u) = \det \begin{pmatrix} y^1(u) & y^1(v) \\ y^2(u) & y^2(v) \end{pmatrix}$$

where $y^i(u)$ is the evaluation of covector $y^i \in V^*$ on vector $u \in V$.

EXAMPLE 5.9.1 (Connection to signed area). In the (x, y)-plane $V = \mathbb{R}^2$, consider the 2-form

$$\omega = dx \wedge dy \in \bigwedge^2 (V^*).$$

It defines a bilinear function $f_\omega \colon V \times V \to \mathbb{R}$ whose geometric meaning is the signed area (see Section 4.7) of the parallelogram spanned by the pair of vectors u, v.

PROPOSITION 5.9.2. *The value of f_ω on the pair (u, v) depends only on the image $\xi = u \wedge v$ in $\bigwedge^2(V)$.*

PROOF. The proof is immediate from the interpretation of $f_\omega(u, v)$ as a generalized signed area of the parallelogram spanned by u and v. Namely, the antisymmetric form f_ω is proportional to the standard area form $\omega = dx \wedge dy$ since $\bigwedge^2 (V^*)$ is 1-dimensional. Furthermore, $dx \wedge dy$ calculates the signed area $A_{u,v}$ of the parallelogram spanned by u and v, where $u \wedge v = A_{u,v}\, e^1 \wedge e^2 \in \bigwedge^2(\mathbb{R}^2)$. Evaluating at the pair (e_1, e_2) we see that f_ω corresponds to ω. $\square$

REMARK 5.9.3. We will sometimes write $\omega(\xi)$ in place of $f_\omega(u, v)$, where $\xi = u \wedge v \in \bigwedge^2(\mathbb{R}^2)$. Thus by definition,

$$\omega(\xi) = f_\omega(u, v).$$

CHAPTER 6

Norms on forms, Wirtinger inequality

6.1. Norm on 1-forms

One major objective of our course is the proof of Gromov's systolic inequality for complex projective space. To this end, we will need to study certain norms, determined by a Riemannian metric on a manifold M, on the de Rham cohomology groups of M defined below in Section 7.9. We start with a general discussion of norms and their duals.

Let V be a vector space. Given a norm $\| \ \|$ (not necessarily of Euclidean type) on V, there is a natural norm on the dual space V^*, defined as follows. We will denote the new norm $\| \ \|^*$ for the purposes of this section.

Definition 6.1.1. The dual norm $\| \ \|^*$ on V^* is defined for $y \in V^*$ by setting $\|y\|^* = \sup \big\{ y(x) \colon x \in V, \|x\| \le 1 \big\}$.

In other words, we calculate the dual norm of the covector y by maximizing its value over vectors $x \in V$ of norm at most 1.

Remark 6.1.2. By homogeneity, the inequality $\|x\| \le 1$ can be replaced by equality in this definition:

$$\|y\|^* = \sup \left\{ y(x) \colon x \in V, \|x\| = 1 \right\},$$

i.e., the norm of $y \in V^*$ can be calculated over the unit vectors $x \in V$.

Example 6.1.3. Consider the plane $V = \mathbb{R}^2$ endowed with the Euclidean norm $|v|$. If $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$ is an orthonormal basis for V, then the dual basis

$$(dx, dy)$$

for V^* is an orthonormal basis. Indeed, $dx(\frac{\partial}{\partial x}) = 1$, $dx(\frac{\partial}{\partial y}) = 0$, and it follows that $|dx|^* = 1$.

Additional examples will appear in the next section.

6.2. Polar coordinates and dual norms

The Euclidean metric in the plane V defines a norm on the tangent plane by the usual identification of a space and its tangent space at a point.

Definition 6.2.1. The norm in the cotangent plane is the norm dual to that on the tangent plane (in the sense of Section 6.1).

Lemma 6.2.2. *In polar coordinates (r, θ) in $V \setminus \{0\}$, we have*

$$|xdy - ydx|^* = r,$$

and therefore

$$\left|\frac{x}{r}dy - \frac{y}{r}dx\right|^* = 1.$$

Proof. The proof is immediate from the fact that dx and dy are orthonormal. Here we use single bars because in this case the norms are Euclidean. □

Note that θ is undefined at the origin. The pair $(dr, d\theta)$ is a basis for the cotangent plane at every point of $V \setminus \{0\}$. However, the basis is not orthonormal. We will specify the norm of $d\theta$ in Proposition 6.2.4.

Lemma 6.2.3. *We have an identity $r^2 d\theta = xdy - ydx$.*

Proof. We have $\tan\theta = \frac{y}{x}$. Equivalently $x \sin\theta = y\cos\theta$. Differentiating with respect to x, we obtain

$$\sin\theta + x\cos\theta\frac{d\theta}{dx} = \frac{dy}{dx}\cos\theta - y\sin\theta\frac{d\theta}{dx}.$$

Multiplying by rdx we obtain

$$ydx + x^2 d\theta = xdy - y^2 d\theta.$$

Thus $xdy - ydx = (x^2 + y^2)d\theta = r^2 d\theta$. □

Proposition 6.2.4. *In the (r, θ) coordinates on $V \setminus \{0\}$, the cotangent plane at each point admits an orthonormal basis given by the pair of 1-forms $(dr,\, rd\theta)$.*

Proof. By Lemma 6.2.2 and Lemma 6.2.3, we obtain

$$|d\theta|^* = \frac{1}{r}.$$

Thus we obtain a unit-norm 1-form $rd\theta = \frac{x}{r}dy - \frac{y}{r}dx$, proving the proposition. □

6.3. Dual bases and dual lattices in a Euclidean space

Given a normed vector space $(V, \| \, \|)$, we defined the dual norm $\| \, \|^*$ on V^* in Section 6.1. We now consider dual bases in Euclidean space. The discussion is similar to the situation with a pair of dual vector spaces treated in Section 2.6. Thus, let $V = \mathbb{R}^n$ be a Euclidean vector space equipped with an inner product denoted $\langle \, , \, \rangle$.

Definition 6.3.1. Given a basis (x_i) for $(V, \langle \, , \, \rangle)$, its *dual basis* is the unique basis (y_j) for V satisfying

$$\langle x_i, y_j \rangle = \delta_{ij},$$

where δ_{ij} is the Kronecker delta function.

Proposition 6.3.2. *Let (x_i) and (y_j) be a pair of dual bases for $\mathbb{R}^n$. Let $A \in \mathrm{Mat}_{n,n}(\mathbb{R})$ be the matrix formed of the column vectors (x_i). Let B be the matrix formed of the column vectors (y_j). Then $B^t A = I_n$.*

This is immediate from the definition of dual bases and matrix multiplication.

Definition 6.3.3. A *lattice* $L \subseteq \mathbb{R}^n$ is the $\mathbb{Z}$-span of n linearly independent vectors.

Definition 6.3.4. Given a lattice $L \subseteq \mathbb{R}^n$, the *dual lattice* $L^* \subseteq \mathbb{R}^n$ is the lattice

$$L^* = \{ y \in \mathbb{R}^n \colon \forall x \in L, \langle x, y \rangle \in \mathbb{Z} \} .$$

Thus the inner product between a point in a lattice and a point in its dual is by definition always an integer.

Theorem 6.3.5. *Consider a $\mathbb{Z}$-basis (x_i) for a lattice $L \subseteq \mathbb{R}^n$. Consider the basis (y_j) dual to the basis (x_i) in $\mathbb{R}^n$. Then (y_j) is a $\mathbb{Z}$-basis for the dual lattice L^*.*

Corollary 6.3.6. *Let L, L^* be a pair of dual lattices in $\mathbb{R}^n$. In the notation of Proposition 6.3.2, we have $\det(A)\det(B) = 1$, i.e., $\mathrm{vol}(\mathbb{R}^n / L)\,\mathrm{vol}(\mathbb{R}^n / L^*) = 1$.*

This is immediate from Proposition 6.3.2 by the multiplicativity of the determinant. In other words, we have the following corollary.

Corollary 6.3.7. *If L and L^* are dual lattices in $\mathbb{R}^n$ then their covolumes multiply to 1.*

6.4. Shortest nonzero vector in a lattice

Definition 6.4.1. Consider $\mathbb{R}^n$ with its Euclidean norm $|\,|$. Given a lattice $L \subseteq \mathbb{R}^n$, denote by $\lambda_1(L)$ the least length of a nonzero vector in L:

$$\lambda_1(L) = \min\left\{|v| \colon v \in L \setminus \{0\}\right\}.$$

We now consider the case of dimension 1. The following proposition is a special case of Corollary 6.3.7 but it is worth spelling it out explicitly.

Proposition 6.4.2. *Let $L \subseteq \mathbb{R}$ be a lattice and $L^* \subseteq \mathbb{R}$ the lattice dual to L. Then*

$$\lambda_1(L)\,\lambda_1(L^*) = 1. \tag{6.4.1}$$

Proof. Given a lattice $L \subseteq \mathbb{R}$, denote by $\alpha > 0$ its generator, so that $L = \mathbb{Z}\,\alpha$. For an element $y \in \mathbb{R}$ to pair integrally with α, it must be an integer multiple of $\beta = \frac{1}{\alpha}$. Thus

$$L^* = \mathbb{Z}\,\beta$$

and $\lambda_1(L^*) = \beta = \frac{1}{\alpha}$. □

Remark 6.4.3. Note that the relation (6.4.1) does not hold in general for dimension greater than 1. Already in dimension 2, the product $\lambda_1(L^*)\lambda_1(L)$ can be greater than 1, as illustrated by the following result.

Example 6.4.4. For the Eisenstein lattice $L_E \subseteq \mathbb{C}$ spanned by the cube roots of unity, we obtain $\lambda_1(L_E^*)\lambda_1(L_E) = \frac{2}{\sqrt{3}}$.

Some optional related material appears in Section 6.12.1.

6.5. Euclidean norm on k-multivectors and k-forms

We studied the linear algebra of multivectors starting in Section 5.8. We will now study norms on the space of multivectors. There are two distinct natural norms on k-multivectors:

(1) the Eucldean norm;
(2) the comass norm.

We start with the Euclidean norm. The comass norm, which plays a crucial role in the proof of Gromov's stable systolic inequality for complex projective space, will be defined in Section 6.6.

Definition 6.5.1. The Euclidean norm $|\,|$ on the space $\bigwedge^k(\mathbb{R}^n)$ of k-multivectors is defined by declaring the basis

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \colon 1 \leq i_1 < i_2 < \cdots < i_k \leq n\} \tag{6.5.1}$$

of (4.10.1) to be orthonormal.

EXAMPLE 6.5.2. Each simple k-form $e_{i_1} \wedge \ldots \wedge e_{i_k}$ has unit Euclidean norm: $|e_{i_1} \wedge \ldots \wedge e_{i_k}| = 1$. All such forms are are all mutually perpendicular.

EXAMPLE 6.5.3. On Euclidean space $V = \mathbb{R}^{2n}$, consider the 2-form $\alpha = e_1 \wedge e_2 + \ldots + e_{2n-1} \wedge e_{2n}$. Then we have $|\alpha| = \sqrt{n}$.

In the next section, we will define a *different* norm on $\bigwedge^k$ which is not Euclidean in general.

6.6. Comass norm

Let V be an inner product space, for instance $\mathbb{R}^n$. Recall that an exterior form is called *simple* or *decomposable* if it can be expressed as a wedge product of 1-forms; see Section 4.9.

LEMMA 6.6.1. *A simple k-multivector $y \in \bigwedge^k V^*$ of the form $y = y^1 \wedge \cdots \wedge y^k$ can be viewed as a k-linear antisymmetric function f_y on V^k via the formula*

$$f_y(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} sgn(\sigma)\, y^1(x_{\sigma(1)}) \ldots y^k(x_{\sigma(k)})$$

as in (5.8.2).

We will use this identification without designating a special symbol for the antisymmetric function. The comass norm is defined as follows.

DEFINITION 6.6.2. The *comass* norm of a k-linear function is its maximal value on a k-tuple of unit vectors in V.

In formulas, the *comass norm* $\|\omega\|$ of a k-linear function $\omega \in \bigwedge^k(V^*)$ is

$$\|\omega\| = \max\left\{\omega(e_1, \ldots, e_k) \colon e_i \in V, |e_i| = 1 \text{ for } 1 \le i \le k\right\}. \tag{6.6.1}$$

Here $|e|$ denotes the Euclidean norm of the vector $e \in V$.

EXAMPLE 6.6.3. The symplectic form $\alpha = e_1 \wedge e_2 + e_3 \wedge e_4$ on $V = (\mathbb{R}^4)^*$ satisfies $|\alpha| = \sqrt{2}$ and

$$\|\alpha\| = 1. \tag{6.6.2}$$

Hence its comass norm is smaller than its Euclidean norm:

$$\|\alpha\| < |\alpha|.$$

Formula (6.6.2) will be proved in the context of the proof of Wirtinger's inequality; see Lemma 6.8.7 and Section 6.10.

LEMMA 6.6.4. *The comass norm for a simple k-form coincides with the natural Euclidean norm on k-forms.*

PROOF. A k-tuple of 1-forms span a k-dimensional subspace $P \subseteq V$. The k-tuple can be replaced by an orthogonal k-tuple forming a basis for P. This can be done by means of a volume-preserving transformation, by applying the Gram-Schmidt process. Thus a simple k-form is proportional to a cup product of an orthonormal k-tuple. $\square$

LEMMA 6.6.5. *Every form $\omega \in \bigwedge^k(V^*)$ satisfies the inequality $\|\omega\| \leq |\omega|$. If ω is simple then equality is attained.*

PROOF. The spaces $\bigwedge^k(V^*)$ and $\bigwedge^k(V)$ are dual by Theorem 5.8.4. Therefore the Euclidean norm $|\omega|$ of ω is defined by formula

$$|\omega| = \max\left\{\omega(\xi) \colon \xi \in \bigwedge^k(V), |\xi| = 1\right\} \tag{6.6.3}$$

similar to the formula (6.6.1) for comass. The difference is that the maximum in (6.6.3) is taken over *all* k-forms in $\bigwedge^k(V)$ and not merely the simple (decomposable) ones. This proves the inequality. $\square$

EXAMPLE 6.6.6. Since every 1-form on V is simple, we have $|\omega| = \|\omega\|$ for all $\omega \in \bigwedge^1(V^*)$.

EXAMPLE 6.6.7. Let $n = \dim(V)$. Since $\bigwedge^n(V^*)$ is 1-dimensional, every n-form is simple, and we have $|\omega| = \|\omega\|$ for all $\omega \in \bigwedge^n(V^*)$.

6.7. Symplectic form from a complex viewpoint

EXAMPLE 6.7.1. Consider the 2-form $dx \wedge dy$. Its associated antisymmetric bilinear function on $\mathbb{R}^2$ is represented by the antisymmetric matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ already discussed in Example 5.2.3.

This motivates the following definitions. Consider the ν-dimensional complex vector space $\mathbb{C}^\nu$. Denote by Z_j the j-th coordinate

$$Z_j \colon \mathbb{C}^\nu \to \mathbb{C},$$

with the usual decomposition $Z_j = X_j + iY_j$ into the real and imaginary parts. The complex conjugate $\bar{Z}_j$ is by definition $\bar{Z}_j = X_j - iY_j$, for all $j = 1, \ldots, \nu$. Then the wedge product $Z_j \wedge \bar{Z}_j$ can be expressed as follows:

$$Z_j \wedge \bar{Z}_j = (X_j + iY_j) \wedge (X_j - iY_j) = -2iX_j \wedge Y_j = \frac{2}{i} X_j \wedge Y_j,$$

or equivalently

$$\frac{i}{2} Z_j \wedge \bar{Z}_j = X_j \wedge Y_j. \tag{6.7.1}$$

Formula (6.7.1) implies the following.

PROPOSITION 6.7.2 (Symplectic form). *Let $Z_1, \ldots, Z_\nu$ be the coordinate functions in $V = \mathbb{C}^\nu$. Then the standard symplectic 2-form α of* (4.9.1)*, namely $\alpha \in \bigwedge^2(V^*)$, is given by*

$$\alpha = \frac{i}{2} \sum_{j=1}^{\nu} Z_j \wedge \bar{Z}_j. \tag{6.7.2}$$

REMARK 6.7.3. A more traditional way of writing the form is in terms of the real basis, by the expression

$$\alpha = \sum_{j=1}^{\nu} X_j \wedge Y_j = X_1 \wedge Y_1 + \ldots + X_\nu \wedge Y_\nu.$$

The expression (6.7.2), emphasizing the complex structure, is useful for future applications, including the proof of Wirtinger's inequality.

EXAMPLE 6.7.4. With respect to the real coordinates, the corresponding coefficient matrix A_α of α as in Definition 5.2.1 is a block diagonal $2\nu \times 2\nu$ matrix with ν diagonal blocks of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ corresponding to each of the complex coordinates Z_j.

6.8. Hermitian product

Let V be a ν-dimensional vector space over $\mathbb{C}$. Let $H = H(v, w)$ be a Hermitian product on V.

EXAMPLE 6.8.1. The standard Hermitian product on $V = \mathbb{C}^\nu$ is given by

$$H(v, w) = \sum_{j=1}^{\nu} \overline{v^j} w^j \tag{6.8.1}$$

where $v = (v^j)$ and $w = (w^j)$.

REMARK 6.8.2. Here we adopt the convention that a Hermitian product H is complex linear in the *second* variable. There are varying conventions in textbooks regarding this issue. We follow Federer's book [**Fe69**] which uses the convention (6.8.1).

LEMMA 6.8.3. *We have*

$$H(v, w) = \overline{H(w, v)}. \tag{6.8.2}$$

DEFINITION 6.8.4. We consider the real part

$$v \cdot w = \mathrm{Re}(H(v, w))$$

of the Hermitian inner product, where v, w are viewed as vectors in $\mathbb{R}^{2n}$.

Thus, $v \cdot w$ is a scalar product on V viewed as a 2ν-dimensional real vector space. Consider also the imaginary part $\alpha = \alpha(v, w)$ of the Hermitian product, so that

$$H(v, w) = v \cdot w + i\alpha(v, w). \tag{6.8.3}$$

REMARK 6.8.5. The above decomposition is similar to the decomposition

$$\bar{z}z' = (x - iy)(x' + iy') = (xx' + yy') + i(xy' - x'y),$$

whenever $z, z' \in \mathbb{C}$.

LEMMA 6.8.6. *The imaginary part α of the Hermitian product is skew-symmetric and therefore can be viewed as an element $\alpha \in \bigwedge^2(V^*)$, the second exterior power of V^*.*

PROOF. We have $\alpha(w, v) = \mathrm{Im}(H(w, v)) = \mathrm{Im}\left(\overline{H(v, w)}\right)$ by formula (6.8.2). Therefore

$$\alpha(w, v) = -\mathrm{Im}(H(v, w)) = -\alpha(v, w),$$

proving anti-commutativity. $\square$

We will use the Hermitian product to prove the following result.

LEMMA 6.8.7. *The comass of the standard symplectic form α satisfies $\|\alpha\| = 1$. The value of the comass is attained by $\alpha(v, w)$ if and only if $R(v) = w$, where $R : V \to V$ is the rotation given by multiplication by i.*

PROOF. By the previous Lemma 6.8.6, the 2-form α is antisymmetric. Thus it suffices to evaluate α on a 2-vector $\xi = v \wedge w$ (see Remark 5.9.3), where v and w are orthonormal. Thus we can assume that $v \cdot w = 0$. We therefore have

$$H(v, w) = i\alpha(v, w) \tag{6.8.4}$$

and $\alpha(v, w) = -iH(v, w) = H(iv, w)$ by complex conjugate-linearity in the first variable. Since $\alpha(\xi)$ is real, the pairing $\langle \xi, \alpha \rangle = \alpha(\xi)$ can be evaluated as follows using (6.8.4):

$$\alpha(\xi) = \alpha(v, w) = H(iv, w) = \mathrm{Re}(H(iv, w)) = (iv) \cdot w \leq 1 \tag{6.8.5}$$

by the Cauchy-Schwarz inequality. Equality in the Cauchy-Schwarz inequality holds if and only if one has $iv = \pm w$. To eliminate the possibility of a minus sign, note that $\alpha(\xi) = 1$ occurs if and only if $iv = w$. $\square$

More generally, one has the following result for the comass of 2-forms.

PROPOSITION 6.8.8. *Let* $\alpha = \lambda_1 X_1 \wedge Y_1 + \cdots + \lambda_n X_n \wedge Y_n \in \bigwedge^2(\mathbb{R}^{2n})$. *Then* $\|\alpha\| = \max_j |\lambda_j|$.

A proof of a more general result appears in Section 6.12.

6.9. Orthogonal diagonalisation

In this section, we recall some standard material from linear algebra, and use it to analyze the structure of 2-forms.

THEOREM 6.9.1. *Every skew-symmetric real matrix can be orthogonally diagonalized into* 2 *by* 2 *blocks as well as a possible block whose entries are identically zero.*

Alternatively, the theorem can be formulated in terms of endomorphisms as in Theorem 6.9.3 below.

DEFINITION 6.9.2. An endomorphism f of $\mathbb{R}^n$ is *anti-selfadjoint* if

$$(\forall x, y \in \mathbb{R}^n) \quad \langle f(x), y\rangle = -\langle x, f(y)\rangle.$$

The following result is well known.

THEOREM 6.9.3. *Given an anti-selfadjoint endomorphism* f *of* $\mathbb{R}^n$, *there exists an orthogonal decomposition of* $\mathbb{R}^n$ *into subspaces* V_j *invariant under* f, *where* $\dim V_j \leq 2$ *for all* j.

We now revert to using α for an arbitrary 2-form. Recall (Section 5.1) that the rank of a 2-form α is the least possible number of simple (decomposable) 2-forms $\alpha^{(j)}$ occurring in a presentation $\alpha = \sum_j \alpha^{(j)}$.

COROLLARY 6.9.4. *The rank of a* 2*-form equals half the rank of the matrix representing the antisymmetric bilinear function.*

PROOF. A 2-form is given by a skew-symmetric matrix A, which can be thought of as an anti-selfadjoint endomorphism $f_A \colon \mathbb{R}^n \to \mathbb{R}^n$.

Step 1. Applying the diagonalisation theorem to f_A, we obtain an orthonormal basis $e_1, \ldots, e_n$ with respect to which the endomorphism decomposes into

(1) μ nonvanishing blocks of size 2 by 2, and
(2) an $s \times s$ block on which the endomorphism vanishes,

where $2\mu + s = n$.

Step 2. With respect to such a basis, the 2-form will have the following form. We have an orthonormal set $\omega_1, \ldots, \omega_{2\mu} \in V^*$ and nonnegative numbers $\lambda_1, \ldots, \lambda_\mu$, so that

$$f_A = \sum_{j=1}^{\mu} \lambda_j \left(\omega_{2j-1} \wedge \omega_{2j}\right). \tag{6.9.1}$$

Therefore $\mathrm{rank}(A) = 2\mu$ whereas the $\mathrm{rank}(\alpha) = \mu$, proving the corollary. □

REMARK 6.9.5. With respect to the new basis, the nonzero part of the matrix will consist of 2 by 2 blocks of the form $\begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix}$.

COROLLARY 6.9.6. *The rank of a 2-form on $V = \mathbb{R}^n$ is at most $\frac{n}{2}$.*

6.10. Wirtinger inequality

We exploit the material of Section 6.9 to prove Wirtinger's inequality for 2-forms. The inequality will be useful in our study of Riemannian metrics on the complex projective space.

Let V be a complex vector space isomorphic to $\mathbb{C}^\nu$. Consider the standard symplectic form $\alpha \in \bigwedge^2(V^*)$ on V. Here $\bigwedge^2(V^*)$ is the space of all 2-linear antisymmetric functions on V (see Theorem 5.8.6). The form α can be thought of as the imaginary part of the standard Hermitian product as in (6.8.3) and (6.10.1).

The comass norm $\| \ \|$ was defined in Definition 6.6.2. Following H. Federer [**Fe69**, p. 40], we prove an optimal upper bound for the comass norm of the exterior powers of a 2-form. We will use the decomposition

$$H(v, w) = v \cdot w + i\,\alpha(v, w) \tag{6.10.1}$$

of a Hermitian inner product into the sum of a scalar product and the symplectic form.

DEFINITION 6.10.1. Let $\mu \le \nu$. We will use the notation $\alpha^{\wedge\mu} = \alpha \wedge \cdots \wedge \alpha$ (μ times).

Thus $\alpha^{\wedge\mu} \in \bigwedge^{2\mu}(V^*)$. We will use the pairing $\langle\ ,\ \rangle$ between dual spaces $\bigwedge(V)$ and $\bigwedge(V^*)$.

THEOREM 6.10.2 (Wirtinger inequality). *Let V be a ν-dimensional complex vector space and α its standard symplectic form. Let $\mu \ge 1$.*

(1) *If $\xi \in \bigwedge^{2\mu} V$ and ξ is simple, then*

$$\langle \xi, \alpha^{\wedge\mu} \rangle \le \mu!\ |\xi|.$$

(2) *equality holds if and only if there exist elements $v_1, \ldots, v_\mu \in V$ such that $\xi = v_1 \wedge (iv_1) \wedge \cdots \wedge v_\mu \wedge (iv_\mu)$.*
(3) *Hence the comass norm satisfies $\|\alpha^{\wedge\mu}\| = \mu!$ for each $\mu \le \nu$.*

We will exploit the following combinatorial result in the proof of Wirtinger's inequality.

Proposition 6.10.3. *In the polynomial* $\left(\sum_{j=1}^{\mu} \lambda_j x_j\right)^{\mu}$ *with commuting variables* x_j*, the coefficient of the monomial* $x_1 x_2 \cdots x_\mu$ *is the product* $\lambda_1 \cdots \lambda_\mu \, \mu!$.

Proof. Let $y_j = \lambda_j x_j$, and consider the product of μ parentheses

$$(y_1 + y_2 + \ldots + y_\mu)\ (y_1 + y_2 + \ldots + y_\mu)\ \cdots\ (y_1 + y_2 + \ldots + y_\mu)\,.$$

This product can be analyzed combinatorially as follows: we have μ possibilities of choosing the variable y_μ out of the μ parenthetical expressions. Out of the remaining $\mu - 1$ parenthetical expressions, we have $\mu - 1$ possibilities for choosing $y_{\mu-1}$. Out of the remaining $\mu - 2$ parenthetical expressions, we have $\mu - 2$ possibilities for choosing $y_{\mu-2}$, etc., resulting in $\mu!$ possibilities altogether. □

Remark 6.10.4. In the calculation of the comass of the powers of the symplectic form, we will apply this combinatorial fact to the commuting variables x_j given by the 2-forms $x_j = \omega_{2j-1} \wedge \omega_{2j}$.

We will prove the special case of Wirtinger inequality when $\mu = \nu$. This case is needed for Gromov's inequality. The general case is treated in Section 6.11.

Theorem 6.10.5. *For any 2-form* α *on* $\mathbb{C}^\mu$*, we have* $\|\alpha^{\wedge\mu}\| \leq \mu!\,\|\alpha\|^\mu$.

Proof of Wirtinger inequality. Let α be a 2-form. By the diagonalisation result of Section 6.9, we can write α as $\alpha = \lambda_1 X_1 \wedge Y_1 + \cdots + \lambda_\mu X_\mu \wedge Y_\mu \in \bigwedge^2(\mathbb{C}^\mu)$. By Proposition 6.8.8, $\|\alpha\| = \max_j |\lambda_j|$. By homogeneity, we can assume $\max_j |\lambda_j| = 1$. Recall that in the top dimension, the Euclidean norm and the comass norm coincide (see Example 6.6.7). By Proposition 6.10.3, we have $\|\alpha\| = |\lambda_1 \cdots \lambda_\mu| \mu! \leq \mu!$, proving the theorem. □

6.11. Proof of Wirtinger inequality in the general case

On the space $V = \mathbb{C}^\nu$, we wish to study the form $\alpha^{\wedge\mu}$ and its evaluation on simple (decomposable) 2μ-multivectors ξ. The main idea is that in real dimension 2μ, every 2-form splits into a sum of at most μ orthogonal simple (decomposable) pieces. Let $|\ |$ be the natural Euclidean norm in $\bigwedge^{2\mu} V$.

Step 1. We can assume that $|\xi| = 1$. To prove Wirtinger's inequality, we need to show that the pairing satisfies the bound $\langle \xi, \alpha^\mu \rangle \leq \mu!$

for all such ξ. The case $\mu = 1$ was treated in Lemma 6.8.7 via the Cauchy–Schwarz inequality.

Step 2. In the general case $\mu \geq 1$, we proceed as follows. Consider the 2μ-dimensional subspace

$$T = T_\xi \subseteq V \tag{6.11.1}$$

associated with the simple 2μ-vector ξ. Here T is spanned by the vectors in the decomposition of ξ as a wedge product of 1-vectors (to be chosen more specifically later). Consider the inclusion map[1] $f\colon T \hookrightarrow V$.

Step 3. Recall that $\alpha \in \bigwedge^2(V^*)$ is (identified with) a bilinear antisymmetric function on V. Consider the restriction of α to the subspace $T \subseteq V$ of (6.11.1). The restriction is denoted $(\wedge^2 f)\alpha \in \bigwedge^2(T^*)$. Let us show that the operations of restriction (to T) and the operation of power-raising are commuting operations.

LEMMA 6.11.1. *The restiction of α^μ to T, denoted $(\wedge^{2\mu} f)\alpha^\mu$, coincides with the μ-th power of the restriction of α to T.*

PROOF. The values of both forms $(\wedge^{2\mu} f)\alpha^{\wedge\mu}$ and $((\wedge^2 f)\alpha)^{\wedge\mu}$ on 2μ-tuples of vectors in T coincide. $\square$

Step 4. We apply Theorem 6.9.3 orthogonally diagonalize the anti-symmetric bilinear function $(\wedge^2 f)\alpha$ on T. Namely, we decompose $(\wedge^2 f)\alpha$ into 2×2 diagonal blocks corresponding to 2-dimensional subspaces of T. Thus, we can choose dual orthonormal bases:

(1) basis $(e_1, \ldots, e_{2\mu})$ of T and
(2) dual basis $(\omega_1, \ldots, \omega_{2\mu})$ of $\bigwedge^1(T^*)$,

and nonnegative numbers $\lambda_1 \geq 0, \ldots, \lambda_\mu \geq 0$, so that

$$(\wedge^2 f)\alpha = \sum_{j=1}^{\mu} \lambda_j \left(\omega_{2j-1} \wedge \omega_{2j}\right). \tag{6.11.2}$$

By Lemma 6.8.7 we have $\|\alpha\| = 1$ and therefore

$$\lambda_j = \alpha(e_{2j-1}, e_{2j}) \leq \|\alpha\| = 1 \tag{6.11.3}$$

for each $j = 1, \ldots, \mu$. By the combinatorial Proposition 6.10.3 (with $x_j = \omega_{2j-1} \wedge \omega_{2j}$) and Lemma 6.11.1 we obtain

$$\left(\wedge^{2\mu} f\right) \alpha^{\wedge\mu} = \mu!\, \lambda_1 \ldots \lambda_\mu\, \omega_1 \wedge \cdots \wedge \omega_{2\mu}. \tag{6.11.4}$$

Step 5. The simple multivector ξ decomposes as $\xi = \epsilon e_1 \wedge \cdots \wedge e_{2\mu}$ with $\epsilon = \pm 1$. Therefore by (6.11.4),

$$\langle \xi, \alpha^{\wedge\mu} \rangle = \epsilon \mu!\, \lambda_1 \ldots \lambda_\mu \leq \mu! \tag{6.11.5}$$

[1] hachala (not hachlala)

since $\lambda_j \leq 1$ from (6.11.3).

Step 6. Equality occurs in (6.11.5) if and only if $\epsilon = 1$ and $\lambda_j = 1$ for each j. Applying the proof of Lemma 6.8.7 based on Cauchy–Schwarz inequality, we conclude that $e_{2j} = R(e_{2j-1})$, for each $j = 1, \ldots, 2\mu$, where $R\colon V \to V$ is the rotation given by multiplication by i. This completes the proof of Wirtinger inequality.

6.12. Wirtinger inequality for an arbitrary 2-form

Recall from Section 3.1 that the polarisation formula allows one to reconstruct a symmetric bilinear form B, from the quadratic form $Q(v) = B(v,v)$ (if the characteristic is not 2):

$$B(v,w) = \tfrac{1}{4}(Q(v+w) - Q(v-w)). \tag{6.12.1}$$

This will be exploited in the proof of Proposition 6.12.1 below.

There is a useful generalisation of Wirtinger's inequality for arbitrary 2-forms. The idea of the proof is still to use the Cauchy-Schwarz inequality, after adjusting the Hermitian product to be compatible with the 2-form under consideration in a suitable sense.

Proposition 6.12.1. *Given an orthonormal basis $(\omega_1, \ldots, \omega_{2\mu})$ for the space $\bigwedge^1(\mathbb{C}^\mu)^*$, and nonzero real numbers $\lambda_1, \ldots, \lambda_\mu$, the 2-form generalizing the symplectic form, $\alpha = \sum_{j=1}^{\mu} \lambda_j\,(\omega_{2j-1} \wedge \omega_{2j})$, has comass $\|\alpha\| = \max_j |\lambda_j|$.*

We can assume without loss of generality that each λ_j is positive. This can be attained in one of two ways. One can permute the coordinates, by applying the transposition flipping ω_{2j-1} and ω_{2j}, so as to change the sign of λ_j. Alternatively, one can replace, say, the basis element ω_{2j} by $-\omega_{2j}$, which similarly changes the sign of λ_j. We now set $\omega'_{2j-1} = \lambda_j^{1/2}\omega_{2j-1}, \quad \omega'_{2j} = \lambda_j^{1/2}\omega_{2j}$. We can then write $\alpha = \sum_{j=1}^{\mu} \omega'_{2j-1} \wedge \omega'_{2j}$. We now exploit the polarisation formula (6.12.1). (The polarisation works on a real slice; one then complexifies to go from a real dot product to a Hermitian product.)

Definition 6.12.2. The modified Hermitian product H_α on $\mathbb{C}^\mu$ is obtained by polarizing the quadratic form Q_α defined by

$$Q_\alpha = \sum_j \left(\left(\omega'_{2j-1}\right)^2 + \left(\omega'_{2j}\right)^2 \right) = \sum_j \left(\left(\lambda_j^{1/2}\omega_{2j-1}\right)^2 + \left(\lambda_j^{1/2}\omega_{2j}\right)^2 \right).$$

Here the squares are understood in the sense of the symmetric product (see (3.1.1)).

LEMMA 6.12.3. *The H_α-norm $|v|_\alpha$ of each H-unit vector v is bounded above by* $(\max_j \lambda_j)^{1/2}$*, namely* $|v|_\alpha \leq \max_j(\lambda_j)^{1/2}$.

PROOF. This is obvious from the definition of the modified Hermitian product. □

PROOF OF PROPOSITION 6.12.1. The Hermitian product H_α is set up in such a way as to satisfy the following property: for each H_α-orthogonal pair v, w, one has $H_\alpha(v,w) = i\alpha(v,w)$, or

$$\alpha(v,w) = -iH_\alpha(v,w). \tag{6.12.2}$$

Now let ζ be a unit 2-vector (with respect to H) such that

$$||\alpha|| = \alpha(\zeta). \tag{6.12.3}$$

Consider its 2-plane $T \subseteq V$. In the 2-plane T, the unit disk of the scalar product $\operatorname{Re} H_\alpha$ is an ellipse with respect to the scalar product $\operatorname{Re} H$. We will exploit the (perpendicular) principal axes of the ellipse. Let v, w be an orthonormal pair proportional to the principal axes. We can then write $\zeta = v \wedge w$. The H-orthonormal pair v, w is also H_α-orthogonal (though in general not H_α-orthonormal), as well. Then, as in (6.8.5) or (6.12.2), we have

$$\alpha(\zeta) = -iH_\alpha(\zeta) = H_\alpha(iv, w) \leq |iv|_\alpha \, |w|_\alpha \leq \max_j \lambda_j$$

by Lemma 6.12.3. □

The result generalizes the the μ-th powers of a 2-form, giving a generalisation of Wirtinger's inequality as stated in the proposition.

COROLLARY 6.12.4. *Every real antisymmetric 2-form A on $\mathbb{C}^\nu$ satisfies the comass bound*

$$\|A^\mu\| \leq \mu! \|A\|^\mu. \tag{6.12.4}$$

PROOF. We may assume that $\|A\| = 1$. By Proposition 6.12.1, we have $\lambda_j \leq 1$ for each $j = 1, \ldots, \nu$.

An inspection of the proof Proposition 6.10.2 reveals that the orthogonal diagonalisation argument (cf. (6.11.3)) applies to an arbitrary 2-form A with comass $\|A\| = 1$. □

6.12.1. $\bigwedge^2(V)$ is isomorphic to the standard model $\bigwedge_0^2(\mathbb{R}^n)$. This material is optional. Let $\bigwedge_0^2(\mathbb{R}^n) = \operatorname{Span}\{e_i \wedge e_j : 1 \leq i < j \leq n\}$, so that one obtains $\dim \bigwedge_0^2(\mathbb{R}^n) = \binom{n}{2}$. Let V be an n-dimensional vector space. As in Section 5.3, we set $F(V) = \operatorname{Span}\left\{e_{(v,w)} : v, w \in V\right\}$.

We define the equivalence relation $\sim$ as in Section 5.3 with the additional relation stipulating that

$$e_{v,w} \sim -e_{w,v}. \tag{6.12.5}$$

We then define

$$\bigwedge^2(V) = F(V)/\sim . \tag{6.12.6}$$

LEMMA 6.12.5. *We have* $\dim \bigwedge^2(V) \leq \binom{n}{2}$.

PROOF. Choose a basis $a_1, \ldots, a_n$ for V. Each $v \in V$ decomposes as a sum $v = v^i a_i$ with $v^i \in \mathbb{R}$. Then

$$e_{(v,w)} = e_{(v^i a_i, w^j a_j)} \sim v^i w^j e_{(a_i,a_j)} \sim \sum_{i<j} v^i w^j e_{(a_i,a_j)} + \sum_{i>j} v^i w^j e_{(a_i,a_j)}.$$

By (6.12.5), we have $e_{(v,w)} \sim \sum_{i<j} v^i w^j e_{(a_i,a_j)} - \sum_{j<i} v^i w^j e_{(a_j,a_i)}$. Switching the dummy indices (i.e., internal summation indices) in the second sum, we obtain $e_{(v,w)} \sim \sum_{i<j} v^i w^j e_{(a_i,a_j)} - \sum_{i<j} v^j w^i e_{(a_i,a_j)}$. Hence

$$e_{(v,w)} \sim \sum_{i<j} (v^i w^j - v^j w^i) e_{(a_i,a_j)}. \tag{6.12.7}$$

This proves the lemma. $\square$

THEOREM 6.12.6. *The model* $\bigwedge^2(V)$ *obtained from the free vector product as in formula* (6.12.6) *is isomorphic to* $\bigwedge_0^2(\mathbb{R}^n)$.

PROOF. We define a map $\zeta\colon \bigwedge^2(V) \to \bigwedge_0^2(\mathbb{R}^n)$ by setting $\zeta(e_{(v,w)}) = \sum_{i<j} (v^i w^j - v^j w^i) e_i \wedge e_j \in \bigwedge_0^2(V)$. In particular, $\zeta\left(e_{(a_i,a_j)}\right) = e_i \wedge e_j \in \bigwedge_0^2(\mathbb{R}^n)$, showing that the map ζ is onto. Combined with Lemma 6.12.5, this proves the theorem. $\square$

CHAPTER 7

Complex projective spaces; de Rham cohomology

7.1. Cell decomposition of real projective space

We start with some examples of decomposing n-dimensional manifolds as union of cells e^i of dimension $i = 0, 1, \ldots, n$.

The discussion is mostly motivational. In particular, some general topology and material on CW-complexes will be assumed without further mention.

Definition 7.1.1 (Cells). Let $n \geq 0$. The n-cell e^n is by definition homeomorphic to an open ball in $\mathbb{R}^n$.[1]

The 0-cell e^0 is a point.

Example 7.1.2. The n-sphere S^n, $n \geq 0$, is the boundary of the $(n+1)$-cell. We have the following examples of cell decompositions of spheres.

(1) The 0-dimensional sphere $S^0 \subseteq \mathbb{R}^1$ consists of two points, ± 1. We can therefore represent it as a disjoint union $S^0 = e^0 \cup a(e^0)$, where $a(x) = -x$ is the antipodal map on $\mathbb{R}$.
(2) The 1-dimensional sphere $S^1 \subseteq \mathbb{R}^2$ can be partitioned into a union $S^1 = e^0 \cup a(e^0) \cup e^1 \cup a(e^1)$, where $e^0 = \{(1,0)\}$ is a point and e^1 is the open upper halfcircle and $a(e^1)$ its antipodal image, where $a\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the antipodal map.
(3) The n-sphere S^n can be decomposed as

$$S^n = e^0 \cup a(e^0) \cup e^1 \cup a(e^1) \cup \ldots \cup e^n \cup a(e^n). \tag{7.1.1}$$

Corollary 7.1.3. *There is a natural chain of inclusions $S^0 \subseteq S^1 \subseteq S^2 \subseteq \cdots \subseteq S^n \subseteq \cdots$ respecting the cell decomposition.*

In Section 1.8 we presented a detailed analysis of the real projective space $\mathbb{RP}^n$ in terms of coordinate charts. For our present purposes, it is convenient to view $\mathbb{RP}^n$ is the quotient of S^n by the antipodal map.

Definition 7.1.4. The real projective space is the space $\mathbb{RP}^n = \{[x] \colon x \in S^n\}$, where $[x]$ is the equivalence class of $x \in S^n$ relative to the equivalence relation $\sim$ defined by $x \sim -x$ for all $x \in S^n$.

[1] Which is homeomorphic to $\mathbb{R}^n$ itself

By decomposition (7.1.1) we obtain the following.

PROPOSITION 7.1.5. *The real projective space admits the following cell decomposition:*

$$\mathbb{RP}^n = e^0 \cup e^1 \cup \ldots \cup e^n, \tag{7.1.2}$$

with a single cell in each dimension $0, \ldots, n$.

EXAMPLE 7.1.6. The projective line $\mathbb{RP}^1 = e^0 \cup e^1$ can be identified with a circle S^1.

The proposition implies the following corollaries.

COROLLARY 7.1.7. *There is a natural chain of inclusions* $\mathbb{RP}^0 \subseteq \mathbb{RP}^1 \subseteq \mathbb{RP}^2 \subseteq \cdots \subseteq \mathbb{RP}^n \subseteq \cdots$

COROLLARY 7.1.8. *The real projective space* $\mathbb{RP}^n$ *can be decomposed as*

$$\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup \mathbb{R}^n, \tag{7.1.3}$$

where $\sqcup$ *denotes disjoint union.*

PROOF. Indeed, an n-cell is homeomorphic to $\mathbb{R}^n$. □

We will specify an analogous decomposition of the complex projective space in Section 7.2.[2]

7.2. Complex projective space $\mathbb{CP}^n$

The real projective space $\mathbb{RP}^n$ was already defined in Section 1.8 and Definition 7.1.4. The complex projective space $\mathbb{CP}^n$ is defined similarly in Definition 7.2.1 below.[3]

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[2]For the complex projective line $\mathbb{CP}^1$, one notes the following. In modern projective geometry, one starts with an arbitrary field K. The familiar construction of "adding a point ∞ at infinity" then produces the projective line $K\mathbb{P}^1 = K \cup \{\infty\}$ over K. For example, when the field $K = \mathbb{C}$ is that of the complex numbers, we thereby obtain the 1-point compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$, namely the Riemann sphere S^2:

$$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} = S^2. \tag{7.1.4}$$

The familiar stereographic projection provides an identification of the complement of a point in S^2 and the complex numbers $\mathbb{C}$.

[3]Generalizing (7.1.4), we will show in Theorem 7.3.2 that one has the decomposition $\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{CP}^{n-1}$ similar to (7.1.3), where $\mathbb{CP}^{n-1}$ is thought of as the hyperplane at infinity. This point of view is explained (in the case $n = 2$) in more detail in the course 88537; see choveret at `http://u.math.biu.ac.il/~katzmik/88-537.html` for the Hartshorne reference.

Definition 7.2.1. A point in $\mathbb{CP}^n$ is represented by $n+1$ complex coordinates as

$$(z^0, z^1, \ldots, z^n) \in \mathbb{C}^{n+1}, \qquad (z^0, z^1, \ldots, z^n) \neq (0, 0, \ldots, 0)$$

where we identify the tuples differing by an overall rescaling:

$$\forall \lambda \in \mathbb{C} \setminus \{0\}, \quad (z^0, z^1, \ldots, z^n) \sim (\lambda z^0, \lambda z^1, \ldots, \lambda z^n). \tag{7.2.1}$$

These are homogeneous coordinates in the traditional sense of projective geometry.

Definition 7.2.2. The homogeneous coordinates of a point in $\mathbb{CP}^n$ are denoted

$$[z] = [z^0, z^1, \ldots, z^n]$$

using the traditional square brackets.

Theorem 7.2.3. *Complex projective n-space $\mathbb{CP}^n$ is a complex manifold of real dimension $2n$. For every point $p \in \mathbb{CP}^n$, there is a natural endomorphism J of $T_p\mathbb{CP}^n$ such as $J^2 = -Id$.*

Proof. For $k = 0, \ldots, n$, we define coordinate charts A_k as in Section 1.8 by setting

$$A_k = \{[z] \colon z^k \neq 0\}. \tag{7.2.2}$$

We define $u_k \colon A_k \to \mathbb{C}^n$ by

$$u_k(z) = \left(\frac{z^0}{z^k}, \ldots, \frac{z^{k-1}}{z^k}, \frac{z^{k+1}}{z^k}, \ldots, \frac{z^n}{z^k} \right). \tag{7.2.3}$$

A given point $p \in A_k$ is the fixed point of the map

$$z \mapsto i(z - p) + p. \tag{7.2.4}$$

The map (7.2.4) induces an endorphism J of T_pA_k such that $J^2 = -\mathrm{Id}$. The transition maps as given by the same formulas as in (1.8.5). Since the transition maps are rational functions and in particular complex-analytic, they commute with J. This gives a well-defined endomorphism of the tangent space T_p at every point of $\mathbb{CP}^n$.[4] □

[4] We can think of both $\mathbb{RP}^n$ and $\mathbb{CP}^n$ as quotients by the action of a compact group as follows. To view the real projective space as a quotient, note that the real projective space can be thought of as an antipodal quotient of the sphere: $\mathbb{RP}^n = S^n/G$, where $G = \{\pm 1\}$ is the cyclic group of order 2. There is a related presentation of complex projective space, where in place of the cyclic group G, we have the unitary group $U(1)$. Recall that the unitary group $U(n)$ is the group of complex $n \times n$ matrices M satisfying the relation $M\bar{M}^t = I_n$. For instance, the group $U(1)$ is the circle of complex numbers of unit absolute value, $S^1 \subseteq \mathbb{C}$. We exploit the action of $U(1)$ on $\mathbb{C}^{n+1}$ by scalar multiplication as in (7.2.1) as follows. We consider the unit sphere $S^{2n+1} \subseteq \mathbb{C}^{n+1}$. The space $\mathbb{CP}^n$ is a quotient space

7.3. Flags and cell decomposition of $\mathbb{CP}^n$

Recall that the real projective space has a cell decomposition with a cell in each dimension; see formula (7.1.2). An analogous decomposition, but with cells only in *even* dimensions, exists for the complex projective space.

DEFINITION 7.3.1. A *complete flag*[5] (V_j) in $V = \mathbb{C}^{n+1}$ is a choice of a nested[6] sequence of subspaces V_j of dimensions $j = 0, 1, 2, \ldots, n+1$, namely:

$$\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n+1} = V.$$

THEOREM 7.3.2. *A choice of a complete flag (V_j) determines a unique decomposition of $\mathbb{CP}^n$ into cells in each even dimension between* 0 *and* $2n$*:*

$$\mathbb{CP}^n = e^0 \sqcup e^2 \sqcup \cdots \sqcup e^{2n} = \mathbb{CP}^{n-1} \sqcup \mathbb{C}^n. \tag{7.3.1}$$

PROOF. The $2j$-dimensional cell $e^{2j} \subseteq \mathbb{CP}^n$ consists of points with homogeneous coordinates $[z^0, z^1, \ldots, z^n]$ contained in the j-th set-theoretic difference: $(z^0, z^1, \ldots, z^n) \in V_j \setminus V_{j-1}$. For more details see Proposition 7.3.4.[7] $\square$

For the standard complete flag (see below), we will represent a point of e^{2j} uniquely by the tuple $(z^0, z^1, \ldots, z^{j-1}, 1)$.

DEFINITION 7.3.3. The *standard complete flag* is the nested sequence

$$\{0\} = V_0 \subseteq \mathbb{C}^1 \subseteq \mathbb{C}^2 \subseteq \cdots \subseteq \mathbb{C}^{n+1},$$

where $V_j = \mathbb{C}^j$ is the subspace of points whose first j coordinates are arbitrary and the remaining $n+1-j$ coordinates vanish.

---

of S^{2n+1} under the action of $U(1)$: $\mathbb{CP}^n = S^{2n+1}/U(1)$. Indeed, every complex line $\mathbb{C}v$ in $\mathbb{C}^{n+1}$ intersects the unit sphere in a circle. We restrict the vectors in the equivalence relation (7.2.1) to the unit sphere, and the scalars λ in (7.2.1) to the unit circle in $\mathbb{C}$. We then identify points in an orbit under the natural action of $U(1)$ by scalar matrices, to obtain $\mathbb{CP}^n$. The *Hopf fibration* is the associated continuous map $S^{2n+1} \to \mathbb{CP}^n$, namely a bundle with fiber S^1, called the Hopf fibration. For $n = 1$, this construction yields the classical Hopf bundle: $S^2 = S^3/U(1)$, where $S^2 = \mathbb{CP}^1$.

[5]degel maleh

[6]mekunenet

[7]Since a $2k$-dimensional cell is homeomorphic to $\mathbb{C}^k$, decomposition (7.3.1) can also be written as follows: $\mathbb{CP}^n = pt \cup \mathbb{C}^1 \cup \cdots \cup \mathbb{C}^n$, or more suggestively as $\frac{\mathbb{C}^{n+1} - pt}{\mathbb{C} - pt} = pt + \mathbb{C}^1 + \mathbb{C}^2 + \ldots + \mathbb{C}^n$. See `https://mathoverflow.net/a/38885` for a discussion of this identity.

With respect to the standard complete flag, we obtain

$$e^{2j} = \big\{[z^0, z^1, \ldots, z^j, 0, \ldots, 0] \colon z^j \neq 0\big\}. \tag{7.3.2}$$

PROPOSITION 7.3.4. *The collection e^{2j} of (7.3.2) of points with coordinates in $V_j \setminus V_{j-1}$ is diffeomorphic to $\mathbb{C}^j$.*

PROOF. The diffeomorphism is

$$\begin{aligned} e^{2j} \to &\quad \mathbb{C}^j \\ (z^0, \ldots, z^{j-1}, z^j, 0, \ldots, 0) \mapsto &\quad \left(\frac{z^0}{z^j}, \ldots, \frac{z^{j-1}}{z^j}\right) \in \mathbb{C}^j, \end{aligned}$$

proving the proposition. $\square$

7.4. Closure of cells produces compact submanifolds

Recall that we have a cell decomposition $\mathbb{CP}^n = e^0 \cup e^2 \cup \ldots \cup e^{2n}$, where for each $j = 0, 1, 2, \ldots, n$, one has in homogeneous coordinates $e^{2j} = \big\{[z^0, z^1, \ldots, z^j, 0, \ldots, 0] \colon z^j \neq 0\big\}$ as in (7.3.2).

THEOREM 7.4.1. *The closure $\overline{e^{2j}} \subseteq \mathbb{CP}^n$ of each cell e^{2j} is the complex projective subspace $\mathbb{CP}^j \subseteq \mathbb{CP}^n$:*

$$\overline{e^{2j}} = \mathbb{CP}^j.$$

PROOF. Recall that we have a decomposition $\mathbb{CP}^j = \mathbb{C}^j \sqcup \mathbb{CP}^{j-1}$. We need to show that an arbitrary point of the projective hyperplane $\mathbb{CP}^{j-1} \subseteq \mathbb{CP}^n$ can be approximated by points in $\mathbb{C}^j$. An arbitrary point of the submanifold $\mathbb{CP}^{j-1}$ has homogeneous coordinates

$$(z^0, \ldots, z^{j-1}, 0, \ldots, 0), \tag{7.4.1}$$

with vanishing coordinates $z^j = \ldots = z^n = 0$. Such a point can be approximated by a point that lies in the cell e^{2j} because it has a non-vanishing j-th coordinate, of the form

$$(z^0, \ldots, z^{j-1}, \epsilon, 0, \ldots, 0),$$

for arbitrarily small $\epsilon \neq 0$. As ϵ tends to zero, we obtain that every point of $\mathbb{CP}^{j-1}$ represented by (7.4.1) is in the closure of the $2j$-cell. Thus the closure is precisely

$$\overline{e^{2j}} = \mathbb{C}^j \cup \mathbb{CP}^{j-1} = \mathbb{CP}^j,$$

completing the proof. $\square$

REMARK 7.4.2. The submanifolds of $\mathbb{CP}^n$ given by $\mathbb{CP}^j$ for $j = 0, 1, 2, \ldots, n$ generate all of the nontrivial homology groups of $\mathbb{CP}^n$; see Section 9.3.

7.4.1. Preliminaries. The material in this section is optional. In this section, we describe a method of transforming a metric $\mathbf{g}$ [8] into a 2-form α on a complex vector space. As motivation, we consider the trigonometric relation

$$\sin(\theta) = \cos(\tfrac{\pi}{2} - \theta). \tag{7.4.2}$$

The corresponding relation in terms of scalar products and determinants is the following.

EXAMPLE 7.4.3. Let $v, w \in \mathbb{R}^2$. If v, w are unit vectors forming an angle θ then

(1) $v \cdot w = \cos\theta$,
(2) $\sin\theta$ is the signed area $\det[v \ \ w]$ of the parallelogram spanned by v, w.

The rotation by $\frac{\pi}{2}$ implied in passing between sine and cosine in formula (7.4.2) can be formalized as follows.

LEMMA 7.4.4. *For any pair $v, w \in \mathbb{C} = \mathbb{R}^2$ of (not necessarily unit) vectors, the 2×2 determinant satisfies*

$$\det[v\ w] = (iv) \cdot w. \tag{7.4.3}$$

The relation (7.4.3) is thought of as analogous to the trigonometric relation (7.4.2). We have the following reformulation.

COROLLARY 7.4.5. *In $\mathbb{C} = \mathbb{R}^2$, the area form and the metric are related by the formula*

$$\alpha(v, w) = \mathbf{g}(iv, w), \tag{7.4.4}$$

where α is the standard symplectic form $dx \wedge dy$ (in this case the area form of $\mathbb{R}^2$), and $\mathbf{g}$ is the standard flat metric $dx^2 + dy^2$ in $\mathbb{C}$.

REMARK 7.4.6. The corollary illustrates the passage from a symmetric form, $\mathbf{g}$, to an antisymmetric form, α, using the rotation of one of the vectors by i. We will now deal with a generalisation of this phenomenon to higher dimensions.

7.4.2. g, α, and J in a complex vector space.

DEFINITION 7.4.7. Let V be a complex vector space. The *complex structure* J on V is the endomorphism

$$J \colon V \to V$$

defined by multiplication by the scalar $i \in \mathbb{C}$, and satisfying $J^2 = -\mathrm{Id}_V$.

EXAMPLE 7.4.8. For $V = \mathbb{C}^\mu$, we have in coordinates

$$J(v) = (iv_1, iv_2, \ldots, iv_\mu).$$

[8] The historical reasons why metrics are denoted by g are discussed at `https://hsm.stackexchange.com/q/3435`.

7.5. Relation to Hermitian product

In Section 6.8 we saw that a Hermitian product $H(v, w)$ on a complex vector space V decomposes as a sum

$$H(v, w) = \mathbf{g}(v, w) + i\alpha(v, w),$$

where $\mathbf{g}$ is a scalar product in $\mathbb{C}^\mu = \mathbb{R}^{2\mu}$ and α is the symplectic form. Here $\mathbf{g}$ is symmetric in the two variables while α is antisymmetric. Moreover, we have the following relation among J, $\mathbf{g}$, and α.

PROPOSITION 7.5.1. *One has the following relation on V:*

$$\alpha(v, w) = \mathbf{g}(Jv, w). \tag{7.5.1}$$

In the special case $V = \mathbb{C}$ the metric $g = dx^2 + dy^2$ and the 2-form $\alpha = dx \wedge dy$ are related by (7.5.1).

PROOF. We have

$$\begin{aligned}
\mathbf{g}(iv, w) &= \mathrm{Re}(H(iv, w)) \quad \text{(by definition)} \\
&= \mathrm{Re}(-iH(v, w)) \quad \text{(skew-linearity)} \\
&= \mathrm{Re}\left(-i(\mathbf{g}(v, w) + i\alpha(v, w))\right) \quad \text{(by definition)} \\
&= \mathrm{Re}(-i(i\alpha(v, w))) \quad \text{(imaginary part of } \mathbf{g}(v, w) \text{ is zero)} \\
&= -i^2\alpha(v, w) \quad (\alpha \text{ is real-valued}) \\
&= \alpha(v, w),
\end{aligned}$$

proving the proposition. □

REMARK 7.5.2. Proposition 7.5.1 enables us, starting with a metric on a complex manifold M, to construct a differential 2-form on M; see Section 7.6. When M is the complex projective space, the 2-form is the closed symplectic form called the Fubini–Study form.

7.6. Explicit formula for Fubini–Study 2-form on $\mathbb{CP}^1$

We will now examine in detail the case of the complex projective line $\mathbb{CP}^1$, i.e. the 2-sphere. We obtain the differential 2-form on $\mathbb{CP}^1$ from the metric by formula (7.5.1). The existence of J is justified by Theorem 7.2.3.

THEOREM 7.6.1. *The round metric of the sphere can be expressed in coordinates given by stereographic projection as*

$$\mathbf{g} = \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2} \tag{7.6.1}$$

in the complement of a point in S^2. This normalisation results in a metric of a round sphere (missing a point) of radius $\frac{1}{2}$ and constant Gaussian curvature $K = +4$.

Proof. The Gaussian curvature K of the metric (7.6.1) is given by $K = -\Delta_{LB} \ln f$ where $f(x,y) = \frac{1}{1+x^2+y^2}$ by formula (3.4.2). We have $\ln f = -\ln(1+x^2+y^2)$ and similarly for y. Then $\frac{\partial}{\partial x}(\ln f) = -\frac{2x}{1+x^2+y^2}$ and

$$\frac{\partial^2}{\partial x^2}(\ln f) = -\frac{2(1+x^2-y^2)}{(1+x^2+y^2)^2}, \quad \frac{\partial^2}{\partial y^2}(\ln f) = -\frac{2(1-x^2+y^2)}{(1+x^2+y^2)^2}. \quad (7.6.2)$$

Adding the formulas in (7.6.2), we obtain $\Delta_{LB} \ln f = -\frac{4}{f^2(1+x^2+y^2)^2} = -4$ and therefore $K = +4$ at every point of the coordinate chart. $\square$

Definition 7.6.2. The *Fubini–Study* 2*-form* α_{FS} on $\mathbb{CP}^1$ is the area form of the metric.

Corollary 7.6.3. *In an affine neighborhood in $\mathbb{CP}^1$ with coordinates (x, y), we have the following formulas for the metric $\mathbf{g}$ and the corresponding Fubini–Study 2-form α_{FS} (the area form of the metric) on the complex projective line: $\mathbf{g} = \frac{dx^2+dy^2}{(1+x^2+y^2)^2}$ and*

$$\alpha_{FS} = \frac{dx \wedge dy}{\left(1+x^2+y^2\right)^2} \quad (7.6.3)$$

Proof. This follows from formula (7.5.1). $\square$

An alternative presentation of the area form on the unit sphere in $\mathbb{R}^3$ in terms of spherical coordinates is the following.

Theorem 7.6.4. *The area form of the 2-sphere of Gaussian curvature $K = +1$ is $\sin\phi \, d\phi \, d\theta$.*[9]

Remark 7.6.5. While formula (7.6.3) applies only in the affine neighborhood, both tensors $\mathbf{g}$ and α_{FS} are globally defined on all of $\mathbb{CP}^1$. This is immediate from (7.4.4); see also (9.8.2).

Remark 7.6.6. The generalisation to $\mathbb{CP}^n$ appears in Section 8.1.

[9]Sometimes the expression $\sin\phi \, d\theta \, d\phi$ is used instead, but this represents the opposite orientation. The theorem was proved in 88-201.

7.7. Exterior differential complex revisited

We defined the exterior differential complex in Section 5.7. Recall that on an n-dimensional differentiable manifold M, we have an exterior differential complex

$$0 \to C^\infty(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \dots \xrightarrow{d_{n-1}} \Omega^n(M) \to 0, \quad (7.7.1)$$

where each $\Omega^i(M)$ is the space of sections of the exterior bundle $\bigwedge^i(T^*M)$.[10]

The following proposition was proved in Section 5.7.

Proposition 7.7.1. *The sequence of maps as in* (7.7.1) *satisfies* $d \circ d = 0$*, or more explicitly* $(\forall k)\ d_k \circ d_{k-1} = 0$*.*

The condition $d \circ d = 0$ is sometimes written as $d^2 = 0$ where d^2 is understood as the composition of two consecutive differentials.[11] We can restate the proposition as follows.

Corollary 7.7.2. *We have an inclusion* $\mathrm{Image}(d_{k-1}) \subseteq \mathrm{Ker}(d_k)$ *in* $\Omega^k(M)$*.*

This inclusion leads to the definition of the de Rham cohomology of M in Section 7.8.

7.8. De Rham cocycles and coboundaries

Let M be a differentiable manifold.

Definition 7.8.1. The group $Z^k_{\mathrm{dR}}(M) = \mathrm{Ker}(d_k) \subseteq \Omega^k(M)$ is called the group of de Rham k-cocycles.

[10] Recall also that the differential $d_i : \Omega^i \to \Omega^{i+1}$ raising the degree by 1, is defined as follows. For $f \in C^\infty(M)$, we define the 1-form $df \in \Omega^1(M)$ by setting $df = d_0 f = \frac{\partial f}{\partial u^j} du^j$. Similarly, the differential $d = d_1 \colon \Omega^1(M) \to \Omega^2(M)$ is defined on a 1-form $f du$ by

$$d(f du) = df \wedge du, \quad (7.7.2)$$

and similarly for the higher differentials.

[11] The following calculation already appeared in Section 5.7. We check the condition $d^2 = 0$ at the level of $\Omega^1(M)$. Given a function $f \in C^\infty(M)$, we first apply the differential $d = d_0$ to obtain a differential 1-form $df = \frac{\partial f}{\partial u^i} du^i$. Next, we apply the differential $d = d_1 \colon \Omega^1(M) \to \Omega^2(M)$ to the 1-form $\frac{\partial f}{\partial u^i} du^i$. Using (7.7.2), we obtain $d\left(\frac{\partial f}{\partial u^i} du^i\right) = d\left(\frac{\partial f}{\partial u^i}\right) du^i = \frac{\partial^2 f}{\partial u^i \partial u^j} du^j \wedge du^i$, with a double summation with indices i and j running from 1 to n independently of each other. Since $du^i \wedge du^i = 0$, we exploit the equality of mixed partials to write $d^2 f = d\left(\frac{\partial f}{\partial u^i} du^i\right) = \sum_{i<j} \frac{\partial^2 f}{\partial u^i \partial u^j}\left(du^j \wedge du^i + du^i \wedge du^j\right) = 0$, since exterior 1-forms anti-commute by definition. Similarly, for a typical 1-form $f du^1$, we have $d(f du^1 \wedge du^2) = \frac{\partial f}{\partial u^i} du^i \wedge du^1$. Therefore $d^2(f du^1 \wedge du^2) = d\left(\frac{\partial f}{\partial u^i} du^i \wedge du^1\right)$. Thus, $d^2(f du^1 \wedge du^2) = \sum_{i<j} \frac{\partial^2 f}{\partial u^i \partial u^j}\left(du^j \wedge du^i + du^i \wedge du^j\right) \wedge u^1 = 0$ as before.

Definition 7.8.2. A differential form ω is called *closed* if $d\omega = 0$.

Thus, a de Rham k-cocycle is by definition a closed differential k-form.

Definition 7.8.3. The group $B^k_{\mathrm{dR}}(M) = \mathrm{Image}(d_{k-1}) \subseteq \Omega^k(M)$ is called the group of de Rham k-coboundaries.

Definition 7.8.4. An *exact differential form* is a de Rham coboundary.[12]

7.9. De Rham cohomology of M, Betti numbers

Definition 7.9.1. The k-th *de Rham cohomology group* of M, denoted $H^k_{\mathrm{dR}}(M)$, is the group

$$H^k_{\mathrm{dR}}(M) = Z^k_{\mathrm{dR}}(M)/B^k_{\mathrm{dR}}(M) = \mathrm{Ker}(d_k)/\mathrm{Image}(d_{k-1})$$

(cocycles modulo the coboundaries).

Even though the object $H^k_{\mathrm{dR}}(M)$ is traditionally referred to as a group, it is in fact a real vector space.

Definition 7.9.2. The k-th *Betti number* $b_k(M)$ of M is the dimension of the real vector space $H^k_{\mathrm{dR}}(M)$: $b_k(M) = \dim H^k_{\mathrm{dR}}(M)$.

As for differential forms (see Theorem 5.6.3), there is a well-defined pullback map.

Theorem 7.9.3. *A differentiable map $f\colon M \to N$ induces a natural "pullback" homomorphism $f^*\colon H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$ for each k.*

The map is defined on a cohomology class in N by pulling back a representative k-form as in Theorem 5.6.3,

7.10. Künneth formula

The Künneth formula enables us to compute the cohomology of a product of two manifolds from the cohomology groups of the two factors.

Theorem 7.10.1 (Kunneth formula). *Let M and N be differentiable manifolds. Then*

$$H^k(M \times N) = \oplus_{i,j,\, i+j=k} H^i(M) \otimes H^j(N). \tag{7.10.1}$$

See Bott and Tu [**BT**, p. 47].

Corollary 7.10.2. *If M, N are connected, then the first Betti number is additive: $b_1(M \times N) = b_1(M) + b_1(N)$.*

[12] Boundary is safa. Coboundary is ko-safa.

Corollary 7.10.3. *If one of M, N is simply connected, then the second Betti number is additive:* $b_2(M \times N) = b_2(M) + b_2(N)$.

7.11. De Rham cohomology in dimensions 0 and n

The de Rham cohomology groups were defined in Section 7.9. We will present several cases where the de Rham cohomology groups can be calculated explicitly.

Proposition 7.11.1. *Every connected manifold M has unit 0-th Betti number $b_0(M) = 1$, i.e.* $H^0_{\mathrm{dR}}(M) \simeq \mathbb{R}$.

Proof. We consider the segment $0 \to C^\infty(M) \to \Omega^1(M)$ of the exterior differential complex.

Step 1. The space of the 0-coboundaries is trivial. The space of 0-cocycles consists of all functions $f \in C^\infty(M)$ satisfing $df = 0$. Thus at every point, all the partial derivatives of a function $f \in Z^0_{\mathrm{dR}}(M)$ must vanish. By the mean value theorem, such a function must be locally constant.

Step 2. Since M is connected, the function f must also be globally constant.

Step 3. We therefore obtain an identification $Z^0_{\mathrm{dR}}(M) = H^0_{\mathrm{dR}}(M) \simeq \mathbb{R} \subseteq C^\infty(M)$ of the 0-th de Rham cohomology group with the space of constant functions on M. $\square$

A similar argument shows the following.

Corollary 7.11.2. *In general $H^0(M) \simeq \mathbb{R}^{|\pi_0(M)|}$ where $\pi_0(M)$ is the set of connected components of M.*

Definition 7.11.3 (Closed manifold). A manifold M is called *closed* if it is compact without boundary.

Definition 7.11.4 (Orientable manifold). M is called *orientable* if it admits an atlas where the determinant of the Jacobian matrix of each transition function ϕ (see Definition 1.1.3) is positive everywhere: $\det(J_\phi) > 0$.

Theorem 7.11.5. *Every closed connected orientable n-dimensional manifold M satisfies $b_n(M) = 1$, i.e.,* $H^n_{\mathrm{dR}}(M) \simeq \mathbb{R}$.

The proof of this result is more difficult than the previous one. We will establish it in some special cases in Sections 7.12 and 7.14.

7.12. de Rham cohomology of a circle

THEOREM 7.12.1. *We have an isomorphism* $H^1_{\mathrm{dR}}(S^1) \simeq \mathbb{R}$*, i.e.,* $b_1(S^1) = 1$*. The isomorphism is given by sending the class* $[\omega]$ *to the real number* $\Phi(\omega) = \oint_{S^1} \omega$*.*

PROOF. For the 1-form $d\theta$ on the unit circle, we have $\oint_{S^1} d\theta = 2\pi$, showing that Φ is surjective. It therefore suffices to show that the kernel of Φ coincides with exact forms. Any smooth 1-form ω can be written as $f(\theta)d\theta$ where f is a smooth function on the circle. We have $\oint_{S^1} \omega = \int_0^{2\pi} f(\theta)d\theta = F(2\pi) - F(0)$ for a primitive function F on $[0, 2\pi]$. If ω is in the kernel then $F(2\pi) = F(0)$. Therefore F extends to a smooth function on the circle, and $\omega = d(Fd\theta)$ as required.[13] □

7.13. Fubini–Study form and the area of $\mathbb{CP}^1$

We will show in Section 7.14.1 that $H^2_{\mathrm{dR}}(\mathbb{CP}^1) \simeq \mathbb{R}$. A specific representative of a non-trivial 2-dimensional cohomology class in $H^2_{\mathrm{dR}}(\mathbb{CP}^1)$, called the Fubini–Study form $\alpha_{FS} \in \Omega^2(\mathbb{CP}^1)$, was defined in Section 7.6 as the bilinear antisymmetric function

$$\alpha_{FS}(u, v) = \mathbf{g}(Ju, v)$$

[13] In more detail, consider the circle $M = S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$. Thus M is a compact connected 1-dimensional manifold (in fact a unique such manifold up to diffeomorphism). The standard 1-form $d\theta$ on M satisfies $\oint_M d\theta = 2\pi$, where $\oint_M$ is the path integral over the circle. It follows by Stokes theorem that it is not exact. The form $d\theta$ is not exact even though it appears to be the d of θ. However, θ is multiple-valued, and a single-valued branch cannot be chosen in a continuous fashion over the entire circle. We will use integration to construct the required isomorphism. Recall that all 1-forms on a 1-dimensional manifold are closed. Consider the space of 1-forms $\Omega^1(M)$. We define a homomorphism $\Phi \colon \Omega^1(M) \to \mathbb{R}$ as follows. Let $\omega \in \Omega^1(M)$ be a 1-form. We define a real number $\Phi(\omega) \in \mathbb{R}$ depending on ω by setting $\Phi(\omega) = \oint_M \omega$. Since $d\theta$ is nonvanishing at every point, we can express ω in terms of the standard 1-form $d\theta \in \Omega^1(M)$ by writing $\omega = f(\theta)d\theta$, where f is a suitable single-valued continuous function on the circle. Thus, $\Phi(\omega) = \int_0^{2\pi} f(\theta)d\theta = F(2\pi) - F(0)$, where $F \colon [0, 2\pi] \to \mathbb{R}$ is an antiderivative for f, so that $dF = f(\theta)d\theta$. Note that in general $F(0) \neq F(2\pi)$. We now show that the 1-form $\omega - \frac{1}{2\pi}\Phi(\omega)d\theta$ is exact. Consider the function $g(\theta) = F(\theta) - \frac{1}{2\pi}\Phi(\omega)\theta$ where $0 \leq \theta < 2\pi$. The function g satisfies $g(2\pi) = F(2\pi) - \frac{1}{2\pi}\Phi(\omega)2\pi = F(2\pi) - F(2\pi) + F(0) = F(0) = g(0)$. Periodicity in θ implies that g descends to a smooth single-valued function on the circle. Therefore by definition, we have $dg \in B^1_{\mathrm{dR}}(M)$. Note that $dg = f(\theta)d\theta - \frac{1}{2\pi}\Phi(\omega)d\theta = \omega - \frac{1}{2\pi}\Phi(\omega)d\theta$. Therefore the kernel $\ker\Phi$ consists precisely of the de Rham 1-coboundaries, i.e., the exact forms. By the isomorphism theorem from basic group theory, the 1-cohomology group of the circle is isomorphic to $\mathbb{R}$, where the equivalence class $[\omega] \in H^1_{\mathrm{dR}}(M)$ of a form $\omega \in \Omega^1(M)$ corresponds to the real number $\Phi(\omega)$.

where $\mathbf{g}$ is a round metric on $\mathbb{CP}^1$ and J is the complex structure. In an affine neighborhood, α_{FS} is given by $\frac{1}{(1+r^2)^2}dx \wedge dy$. The form α_{FS} is a globally defined closed 2-form on $\mathbb{CP}^1$ (in this dimension it can be viewed as the area form of the metric). Its cohomology class

$$[\alpha_{FS}] \in H^2_{\mathrm{dR}}(\mathbb{CP}^1)$$

spans $H^2_{\mathrm{dR}}(\mathbb{CP}^1) \simeq \mathbb{R}$.[14]

THEOREM 7.13.1. *We have* $\operatorname{area}(\mathbb{CP}^1) = \int_{\mathbb{CP}^1} \alpha_{FS} = \pi$.

PROOF. We use polar coordinates and formula (7.6.3) as follows:

$$\begin{aligned}
\operatorname{area}(\mathbb{CP}^1) &= \int_{\mathbb{CP}^1} \alpha_{FS} \\
&= \iint\limits_{\mathbb{C}} \frac{1}{(1+r^2)^2} dx \wedge dy \\
&= \iint \frac{r dr d\theta}{(1+r^2)^2} \\
&= 2\pi \int_0^\infty \frac{r dr}{(1+r^2)^2} \\
&= \pi \int_1^\infty \frac{du}{u^2} \\
&= \pi,
\end{aligned}$$

where we used the u-substitution $u = 1 + r^2$. □

REMARK 7.13.2 (Normalisation of the metric). The resulting value π for the area is consistent with the fact that we are dealing with a 2-sphere of Gaussian curvature 4, radius $\frac{1}{2}$, and Riemannian diameter $\frac{\pi}{2}$; cf. Theorem 7.6.1.[15]

7.14. 2-cohomology group of the torus and sphere

In this section we will compute the top-dimensional de Rham cohomology group of the 2-torus. In Section 7.14.1 we will do the same for the 2-sphere.

[14]In the sequel it will be important that the 2-form α_{FS} has unit comass (see Section 6.6) at every point. A more general formula defines such a 2-form α_{FS} on the projective space $\mathbb{CP}^n$ (see Section 9.9). Its class $[\alpha_{FS}]$ spans the group $H^2_{\mathrm{dR}}(\mathbb{CP}^n)$.

[15]In Section 9.5 we will define an integer lattice $L^2_{\mathrm{dR}}(\mathbb{CP}^n)$ in the de Rham cohomology of $\mathbb{CP}^n$. The element $\left[\frac{1}{\pi}\alpha_{FS}\right] \in H^2_{\mathrm{dR}}(\mathbb{CP}^1)$ is a generator of the integer lattice $L^2_{\mathrm{dR}}(\mathbb{CP}^1) \simeq \mathbb{Z}$.

THEOREM 7.14.1. *We have an isomorphism* $H^2_{\mathrm{dR}}(\mathbb{T}^2) \simeq \mathbb{R}$*, i.e.,* $b_2(\mathbb{T}^2) = 1$*. The isomorphism is given by sending* $[\omega] \in H^2_{\mathrm{dR}}(\mathbb{T}^2)$ *to the real number* $\int_{\mathbb{T}^2} \omega$*.*

PROOF. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the torus obtained as the quotient by the standard integer lattice in the (x, y)-plane. Let x and y be the standard coordinates on the torus, both ranging from 0 to 1. Note that $\mathbb{T}^2$ is obtained from the square $[0,1]\times[0,1]$ by means of the familiar identifications on the boundary. Let $\alpha = dx \wedge dy$ be the area form of the torus.

Step 1. Recall that on a 2-dimensional manifold all 2-forms are closed. Let $\Omega^2(\mathbb{T}^2)$ be the space of 2-forms on the torus. Since the vector space $\bigwedge^2(\mathbb{R}^2)$ is 1-dimensional and α is nonvanishing at every point, each 2-form can be written as $f(x, y)\alpha$ for a suitable a continuous function f on the torus.

Step 2. Consider the map $\Phi \colon \Omega^2(\mathbb{T}^2) \to \mathbb{R}$ given by the integral over the torus, namely $\Phi(f\alpha) = \int_{\mathbb{T}^2} f\alpha$. As in the case of the circle (see Section 7.12) this map will induce the required isomorphism.

Step 3. Suppose $f(x, y)$ is a function satisfying $\Phi(f\alpha) = 0$. To prove the theorem, it suffices to show that the 2-form $f(x, y)\alpha$ is necessarily exact. Then the homomorphism Φ descends to the required isomorphism with $\mathbb{R}$. Thus we need to find functions $g(x, y)$ and $h(x, y)$ on the torus such that

$$f\, dx \wedge dy = d(g\, dx + h\, dy). \tag{7.14.1}$$

In fact h can be chosen to depend on the first variable only.[16]

Step 4. For each x, we consider the average in the y-direction, namely $a(x) = \int_0^1 f(x, t)dt$. We define g by setting

$$g(x, y) = -\int_0^y f(x, t)dt + a(x)y. \tag{7.14.2}$$

Let us show that the function g is periodic in both x and y and therefore is well defined on the torus. We have

$$g(x, 1) = -a(x) + a(x) = 0 = g(x, 0).$$

Hence the function g is periodic in the variable y. Furthermore,

$$a(0) = \int_0^1 f(0, t)dt = \int_0^1 f(1, t)dt = a(1)$$

[16]This feature will be useful when we develop a similar proof for S^2 in Section 7.14.1.

since f is periodic in x. It follows that g is periodic in the x-direction, as well, so that we can write $g \in C^\infty(\mathbb{T}^2)$.

Step 5. We need to choose a function h appropriately so as to satisfy equation (7.14.1) which shows that $f dx \wedge dy$ is exact. We have

$$d(g\,dx + h\,dy) = \frac{\partial g}{\partial y} dy \wedge dx + \frac{\partial h}{\partial x} dx \wedge dy = \left(-\frac{\partial g}{\partial y} + \frac{\partial h}{\partial x} \right) dx \wedge dy$$

by antisymmetry of wedge product. Thus equation (7.14.1) is equivalent to the following PDE:

$$-\frac{\partial g}{\partial y} + \frac{\partial h}{\partial x} = f. \tag{7.14.3}$$

We apply the fundamental theorem of calculus to the formula (7.14.2), to obtain $\frac{\partial g}{\partial y} = -f + a(x)$. Then equation (7.14.3) becomes $\frac{\partial h}{\partial x} = a(x)$ and we set

$$h(x) = \int_0^x a(s) ds.$$

Note that h depends only on x.

Step 6. To show that h descends to a continuous function on the torus, note that

$$h(1) = \int_0^1 a(s) ds = \int_0^1 \int_0^1 f(s,t) dt ds = 0$$

since f has total integral zero by hypothesis (see Step 3 above). Of course $h(0) = 0$ also, so h is a periodic function and therefore $h \in C^\infty(\mathbb{T}^2)$. This establishes the existence of the required functions g and h and the 1-form $g\,dx + h\,dy \in \Omega^1(\mathbb{T}^2)$, proving the theorem. $\square$

7.14.1. 2-cohomology group of the sphere. The material in this section is optional. In Section 7.14, we showed that $b_2(\mathbb{T}^2) = 1$. A modification of the same argument shows that the same holds for the 2-sphere. We use spherical coordinates $(\theta, \phi) \in [0, 2\pi] \times [0, \pi]$, where $z = \cos\phi$, $r = \sin\phi$, $x = r\cos\theta$, $y = r\sin\theta$. The form

$$\alpha = \sin\phi\, d\phi \wedge d\theta \text{ [17]} \tag{7.14.4}$$

is the area form for the unit sphere S^2 by Theorem 7.6.4.

LEMMA 7.14.2. *The following* 1*-forms admit smooth extensions to* S^2*:*

(1) $\sin^2\phi\, d\theta$*, so that we can write* $\sin^2\phi\, d\theta \in \Omega^1(S^2)$*;*
(2) $\sin\phi\, d\phi \in \Omega^1(S^2)$.

[17] See discussion of sign in note 9 of Chapter 7.

PROOF. By Lemma 6.2.3, we have $\sin^2\phi\, d\theta = r^2 d\theta = xdy - ydx$. Thus the 1-form can be thought of as the restriction[18] of the smooth form $xdy-ydx$ from $\mathbb{R}^3$ to S^2.

Similarly, since $z = \cos\phi$, we have $dz = -\sin\phi\, d\phi$. Therefore the form $\sin\phi\, d\phi$ can be thought of as the restriction of $-dz$ from $\mathbb{R}^3$ to S^2, establishing smoothness. $\square$

THEOREM 7.14.3. *We have an isomorphism $H^2_{\mathrm{dR}}(S^2) \simeq \mathbb{R}$, i.e., $b_2(S^2) = 1$. The isomorphism is given by sending $[\omega] \in H^2_{\mathrm{dR}}(S^2)$ to the real number $\int_{S^2} \omega$.*

PROOF. Assume that $f(\theta, \phi) \in C^\infty(S^2)$ has zero average $\int_{S^2} f\alpha = 0$, i.e.,

$$\int_{S^2} f \sin\phi\, d\phi \wedge d\theta = 0. \tag{7.14.5}$$

To prove the theorem, we need to show that such a form $f\sin\phi\, d\phi \wedge d\theta$ is exact. We will solve the equation analogous to (7.14.1), making the necessary changes to allow for the factor $\sin\phi$ in (7.14.4) as follows. We are looking for functions $G(\theta, \phi)$ and $H(\phi)$ on the sphere such that

$$f\, d\theta \wedge \sin\phi\, d\phi = d(Hd\theta + G\, \sin\phi\, d\phi). \tag{7.14.6}$$

By Lemma 7.14.2, the form $\sin\phi\, d\phi$ is a well-defined global 1-form vanishing at the poles. Thus

(1) G can be chosen to be any smooth function;
(2) H needs to be chosen so as to compensate for the singularity of $d\theta$ at the poles.[19]

For each ϕ, we define the average

$$A(\phi) = \frac{1}{2\pi} \int_{\theta=0}^{\theta=2\pi} f(\theta, \phi) d\theta. \tag{7.14.7}$$

Then $A(0)$ equals the value of f at the north pole, while $A(\pi)$, at south pole (these values may be nonzero). Note that

$$\int_{\phi=0}^{\phi=\pi} A(\phi) \sin\phi\; d\phi = 0 \tag{7.14.8}$$

by (7.14.5). We define G by setting

$$G(\theta, \phi) = \int_0^\theta f(t, \phi) dt - A(\phi)\theta \tag{7.14.9}$$

where A is an in (7.14.7). Then

(1) G is 2π-periodic in θ;
(2) $G(\theta, \phi)$ defines a continuous function on S^2.

[18] More precisely, pullback by the inclusion map; see Section 5.6.

[19] It may be possible to incorporate the factor $\sin^2\phi$ in formula 7.14.6 by writing $\sin^2\phi d\theta$ in place of $d\theta$. This may make subsequent verifications for H simpler.

We will choose a function $H(\phi)$ appropriately so as to satisfy equation (7.14.6). We have

$$\begin{aligned} d(H\,d\theta + G\,\sin\phi\,d\phi) &= \frac{dH}{d\phi}\,d\phi\wedge d\theta + \frac{\partial G}{\partial\theta}\,d\theta\wedge\sin\phi\,d\phi \\ &= \Big(-\frac{dH}{d\phi} + \frac{\partial G}{\partial\theta}\sin\phi\Big)\,d\theta\wedge d\phi \end{aligned}$$

by the antisymmetry of wedge product. Thus equation (7.14.6) is equivalent to the following differential equation:

$$-\frac{dH}{d\phi} + \frac{\partial G}{\partial\theta}\sin\phi = f\sin\phi. \tag{7.14.10}$$

We apply the fundamental theorem of calculus to (7.14.9), to obtain $\frac{\partial G}{\partial\theta} = f(\theta,\phi) - A(\phi)$. Then equation (7.14.10) becomes $\frac{dH}{d\phi} = -A(\phi)\sin\phi$. We therefore set

$$H(\phi) = -\int_0^\phi A(s)\sin s\;ds. \tag{7.14.11}$$

Note that $H(0) = 0$. Furthermore, it follows from (7.14.11) that H has a zero of order at least 2 at 0 (the domain of integration $[\phi,\pi]$ tends to zero and sine tends to zero), compensating for the singularity of $d\theta$ at the north pole by Lemma 7.14.2. Meanwhile, $H(\pi) = 0$ from (7.14.8) since f has average zero with respect to the area form of S^2 by hypothesis. Hence $H(\phi)$ defines a function on S^2 which vanishes at both poles. When $\phi \to \pi$ we have $H(\phi) = +\int_\phi^\pi A(s)\sin s\;ds$. Therefore H has a zero of order 2 at the south pole, as well (the domain of integration tends to zero and sine tends to zero). It follows that $H d\theta$ is a well-defined global 1-form. Therefore the 1-form $H d\theta + G\sin\phi\,d\phi$ is a primitive for the 2-form $f\alpha$. It follows that $f\alpha$ is exact, and therefore integration induces an isomorphism between $H^2_{\mathrm{dR}}(S^2)$ and $\mathbb{R}$ as in Section 7.14. $\square$

CHAPTER 8

De Rham cohomology and topology

8.1. Fubini–Study form and volume form on $\mathbb{CP}^n$

In Section 7.6, we expressed the Fubini–Study metric $\mathbf{g}$ and the 2-form α_{FS} on $\mathbb{CP}^1$ explicitly in an affine neighborhood; namely $\mathbf{g} = \frac{dx^2+dy^2}{(1+r^2)^2}$ and $\alpha_{FS} = \frac{dx\wedge dy}{(1+r^2)^2}$. The metric and the 2-form are defined globally on $\mathbb{CP}^1$, and satisfy the relation $\alpha_{FS}(u,v) = \mathbf{g}(Ju,v)$ at every point of $\mathbb{CP}^1$. The metric and the 2-form generalize to higher dimensions as follows.

Theorem 8.1.1. *Let $p,q \in \mathbb{C}^{n+1}$ represent points $\bar{p},\bar{q} \in \mathbb{CP}^n$ in homogeneous coordinates. A metric $\mathbf{g}$ on $\mathbb{CP}^n$ is uniquely determined by the distance function*

$$d(\bar{p},\bar{q}) = \arccos\frac{|H(p,q)|}{|p|\,|q|}, \tag{8.1.1}$$

where H is the Hermitian inner product in $\mathbb{C}^{n+1}$.

For more details see Section 8.

Remark 8.1.2. The formula should be compared to the analogous formula (1.8.7) for the real projective space.

Lemma 8.1.3. *The metric $\mathbf{g}$ on $\mathbb{CP}^n$ is determined by the distance function d of formula* (8.1.1).

Proof. To pass from the metric to the distance function $d(x,y)$, one takes the infimum of integrals $\int |\gamma'(t)|\,dt$ over paths γ joining x and y, where the norm is determined by the metric:

$$d(x,y) = \inf\left\{\int_0^1 |\gamma'(t)|\,dt\colon \gamma(0)=x, \gamma(1)=y\right\}.$$

Conversely, the norm of a tangent vector $v \in T_x$ can be computed as

$$|v| = \lim_{t\to 0}\frac{d(x,\gamma(t))}{t}$$

where γ is a smooth curve such that $\gamma(0)=x$ and $\gamma'(0)=v$. $\square$

The complex structure J in the tangent space at each point of $\mathbb{CP}^n$ is provided by Theorem 7.2.3. We can exploit the complex structure to define the Fubini–Study form as in Section 7.4.2.

DEFINITION 8.1.4. The Fubini–Study form $\alpha_{FS} \in \Omega^2(\mathbb{CP}^n)$ is the differential 2-form $\alpha_{FS}(v,w) = \mathbf{g}(Jv,w)$.

REMARK 8.1.5. We will provide an explicit formula for the Fubini–Study 2-form α_{FS} in an affine neighborhood of $\mathbb{CP}^n$ in Section 9.9.

THEOREM 8.1.6. *The wedge power*

$$\alpha_{FS}{}^{\wedge n}$$

of the Fubini–Study form is a nonzero $2n$-form at every point of $\mathbb{CP}^n$.

PROOF. Both the metric and the 2-form are invariant under unitary transformations. It suffices to check such a relationship in the tangent space at a point such as the origin in an affine neighborhood. The Fubini–Study form at the origin is the symplectic form $\sum_{j=1}^n dx^j \wedge dy^j$. Its top exterior power is

$$n!\, dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n, \tag{8.1.2}$$

as a special case of the calculation peformed in the context of the proof of Wirtinger's inequality in Section 6.10. The form (8.1.2) is nonzero and spans the complex line $\bigwedge^{2n}(\mathbb{C}^n)$. $\square$

8.2. Cup product, ring structure in de Rham cohomology

De Rham cohomology of a manifold M possesses a natural product structure, described as follows. Recall the following:

(1) The exterior algebra $\bigwedge(T_p^*)$ at every point $p \in M$ possesses a wedge product.
(2) The associated complex of differential forms $\Omega(M)$ similarly possesses a product where two differential forms are multiplied pointwise using the wedge product in the exterior algebra.

DEFINITION 8.2.1. The cup-product $\cup$ of cohomology classes is the operation

$$\cup : H^a_{\mathrm{dR}}(M) \times H^b_{\mathrm{dR}}(M) \to H^{a+b}_{\mathrm{dR}}(M) \tag{8.2.1}$$

defined by the wedge product $\wedge$ at the level of the representing differential forms.

The cup product is well-defined due to the following pair of lemmas.

LEMMA 8.2.2. *The wedge product of a pair of closed differential forms is still closed.*

PROOF. This is immediate from the Leibniz rule (5.7.1). Indeed, if forms α and β are both closed, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta \pm \alpha \wedge d\beta = 0,$$

so that $\alpha \wedge \beta$ is closed, as well. □

LEMMA 8.2.3. *The product in cohomology is independent of the choice of representative closed differential forms.*

PROOF. Let α and β be closed forms, and γ an arbitrary form such that the forms α and $d\gamma$ are in the same space $\Omega^k(M)$. Then

$$(\alpha + d\gamma) \wedge \beta = \alpha \wedge \beta + d\gamma \wedge \beta = \alpha \wedge \beta + d(\gamma \wedge \beta)$$

since β is closed. Therefore adding an exact form to α does not change the cohomology class $[\alpha \wedge \beta]$. Thus, given a pair of classes $[\alpha] \in H^a_{\mathrm{dR}}(M)$ and $[\beta] \in H^b_{\mathrm{dR}}(M)$, we obtain a well-defined class

$$[\alpha \cup \beta] \in H^{a+b}_{\mathrm{dR}}(M),$$

as required. □

DEFINITION 8.2.4 (Ring structure). The *de Rham cohomology ring* is the graded ring[1]

$$\left(\oplus_{k=0}^n H^k_{\mathrm{dR}}(M),\ \cup\right)$$

where

(1) the additive structure comes from the real vector space structure on the individual groups, and
(2) the multiplicative structure is given by the cup product $\cup$ of formula (8.2.1).[2]

8.3. Cohomology of complex projective space

The following result generalizes Corollary 7.14.3 on $\mathbb{CP}^1$.

THEOREM 8.3.1. *The Betti numbers of complex projective space $\mathbb{CP}^n$, $n \geq 1$, satisfy*

$$b_k(\mathbb{CP}^n) = \begin{cases} 1 & \text{for all even } \ k = 0, 2, \ldots, 2n \\ 0 & \text{for all other } k. \end{cases}$$

[1]algebra medureget

[2]When the cup product in the de Rham cohomology of M vanishes, one can sometimes use the differential graded associative algebra (dga) $\Omega(M) = \oplus_k \Omega^k(M)$ to define finer invariants called Massey products. See e.g., [**Ka07**] for a summary.

Remark 8.3.2. The geometric counterpart of this result is the decomposition of complex projective space with a single cell in each even dimension between 0 and $2n$ (see (7.3.1)) where the closure of these cells gives precisely the complex projective subspaces $\mathbb{CP}^k$, $k = 0, \ldots, n$.

We have the following analogous fact for 2-dimensional homology (dealt with in Section 9.3) and higher homology of $\mathbb{CP}^n$ (given here for general culture).

Theorem 8.3.3. *The complex submanifolds*

$$\{p\} \subseteq \mathbb{CP}^1 \subseteq \mathbb{CP}^2 \subseteq \mathbb{CP}^3 \subseteq \cdots \subseteq \mathbb{CP}^{n-1} \subseteq \mathbb{CP}^n$$

generate all of the homology of complex projective space $\mathbb{CP}^n$*, so that* $[\mathbb{CP}^k]$ *is a generator of* $H_{2k}(\mathbb{CP}^n; \mathbb{Z})$.[3]

Returning to cohomology, we note that Theorem 8.3.1 can be refined as follows, taking into account the ring structure in cohomology with product operation $\cup$ described in Section 8.2.

Theorem 8.3.4. *The cohomology ring of* $\mathbb{CP}^n$ *is the truncated*[4] *polynomial ring in a single* 2*-dimensional variable. Namely, the ring is generated by a single class* $\omega \in H^2_{\text{dR}}(\mathbb{CP}^n)$*, with the unique relation* $\omega^{\cup(n+1)} = 0$*, where* ω *can be taken to be the class of the Fubini–Study* 2*-form* $\alpha_{FS} \in \Omega^2(\mathbb{CP}^n)$.

Corollary 8.3.5. *Let* $\omega = [\alpha_{FS}]$*. Then the* $2n$*-dimensional class* $\omega^{\cup n} \in H^{2n}_{\text{dR}}(\mathbb{CP}^n)$ *spans the group* $H^{2n}_{\text{dR}}(\mathbb{CP}^n) \simeq \mathbb{R}$.

8.4. Abelianisation in group theory

This section and the ones following contain a review of fundamental groups and homology groups.

Definition 8.4.1. Given a group π defined by generators (g_i) and relations (r_k), one usually writes

$$\pi = \langle g_i \mid r_k \rangle.$$

Example 8.4.2. Two examples:

(1) The group $\mathbb{Z}$ can be presented as follows:

$$\mathbb{Z} = \langle g_1 \mid \varnothing \rangle,$$

where the empty set indicates the absence of relations.

---

[3] For example, the 2-homology class represented by the surface $\mathbb{CP}^1$ is a generator of $H_2(\mathbb{CP}^n; \mathbb{Z}) \simeq \mathbb{Z}$; the 4-homology class represented by the 4-submanifold $\mathbb{CP}^2$ is a generator of $H_4(\mathbb{CP}^n; \mathbb{Z})$ which is similarly isomorphic to $\mathbb{Z}$ for $n \geq 2$.

[4] Katum: kuf, tet, vav, memsofit

(2) The cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\,\mathbb{Z}$ can be presented as follows:
$$\mathbb{Z}/n\,\mathbb{Z} = \langle g_1 \mid g_1^n \rangle,$$
where the right-hand side is written multiplicatively. The multiplicative group can be thought of as the subgroup of $S^1 \subseteq \mathbb{C}$ consisting of the n-th roots of unity, with $g_1 = e^{\frac{2\pi i}{n}}$.

DEFINITION 8.4.3. The *commutator* of two elements $g, h \in \pi$ of a group π is the element
$$[g, h] = ghg^{-1}h^{-1} \in \pi.$$

EXAMPLE 8.4.4. The group $\mathbb{Z}^2$ can be presented as follows in multiplicative notation:
$$\mathbb{Z}^2 = \langle g_1, g_2 \mid [g_1, g_2] \rangle.$$

By augmenting the set of relations of a group π by the commutation relations between each pair of generators, we obtain an abelian group π^{ab} called the abelianisation of π, as follows.

DEFINITION 8.4.5. The *abelianisation* π^{ab} of a group $\pi = \langle g_i \,|\, r_k \rangle$ is the group
$$\pi^{ab} = \left\langle g_i \;\middle|\; r_k, [g_i, g_j]\, \forall i, j \right\rangle$$
obtained by adding a relation given by the commutator of each pair of generators.

LEMMA 8.4.6. *Let $e \in \pi^{ab}$ be the neutral element, and $\bar{g}_i$ the image of g_i under the quotient homomorphism $\pi \to \pi^{ab}$. Then in the abelianisation π^{ab}, the formulas $[\bar{g}_i, \bar{g}_j] = e$ hold.*

8.5. The fundamental group $\pi_1(M)$

The *fundamental group* of a connected manifold M is the group defined as follows. Consider a pair (M, p) where $p \in M$ is a fixed basepoint.

DEFINITION 8.5.1. Let $1 = e^{0i} \in S^1$. A *based loop*[5] in (M, p) is a continuous map $f\colon S^1 \to M$ such that $f(1) = p$.

DEFINITION 8.5.2. We use based loops to define equivalence using homotopy as in introductory algebraic topology[6] to define the fundamental group $\pi_1(M, p)$ as the set of equivalence classes of based loops.

We then have the following.

[5] lul'a mevuseset?

[6] See e.g., Armstrong [**Ar83**].

manifold	π_1
S^1	$\mathbb{Z}$
S^n, $n \geq 2$	$\{1\}$
$\mathbb{CP}^n$	$\{1\}$
$\mathbb{RP}^n$, $n \geq 2$	$\mathbb{Z}_2$
T^n	$\mathbb{Z}^n$

TABLE 8.5.1. Manifolds and their fundamental groups

PROPOSITION 8.5.3. *The fundamental group has the following properties.*

(1) *A based loop $f\colon S^1 \to M$ defines a trivial class in the fundamental group $\pi_1(M,p)$ if and only if the map $S^1 \to M$ can be "filled" by a disk D, i.e., extended to a map $D \to M$ where $\partial D = S^1$.*
(2) *The isomorphism class of the fundamental group $\pi_1(M,p)$ of a connected manifold M is independent of the choice of the basepoint in M.*

The basepoint is frequently suppressed from the notation for the fundamental group, and one writes simply $\pi_1(M)$.

THEOREM 8.5.4. *Five cases of fundamental groups are given in Table* 8.5.1.

The fundamental group of a surface of genus $g \geq 2$ is not abelian. The abelianisation of the fundamental group of a surface of genus g is the group $\mathbb{Z}^{2g}$; see Section 8.12.5 for details.[7]

[7]We provide some remarks on the second homotopy group $\pi_2(M)$. This material is optional. Similarly to the fundamental group $\pi_1(M,p)$, one defines the second homotopy group $\pi_2(M,p)$ as the group generated by based maps $S^2 \to M$ modulo based homotopy connecting a pair of based maps. Similarly to the case of π_1, a map belongs to a trivial class in $\pi_2(M)$ if it can be extended to a map of the 3-ball: $B^3 \to M$. Unlike the fundamental group, the second homotopy group $\pi_2(M)$ is always abelian. The 2-homotopy group of the 2-sphere $S^2 = \mathbb{CP}^1$ is infinite cyclic: $\pi_2(S^2) = \mathbb{Z}$. More generally, the 2-homotopy group of the $\mathbb{CP}^n$ is infinite cyclic: $\pi_2(\mathbb{CP}^n) = \mathbb{Z}$ for each $n = 1, 2, 3, \ldots$ Remark for general culture: The long exact sequence (not defined in this course) of the Hopf fibration helps calculate $\pi_3(S^2) \simeq \mathbb{Z}$. This group is generated by the Hopf map $S^3 \to S^2$ of note 4 of Section 7.

8.6. 1-cycles on manifolds

To formulate Gromov's inequality for $\mathbb{CP}^n$ we will need the notion of the homology group H_k of a manifold M for the values $k = 1, 2$.

The singular homology groups with integer coefficients, $H_k(M; \mathbb{Z})$ for $k = 0, 1, \ldots$ of M are abelian groups which are homotopy invariants of the manifold M. Developing the singular homology theory in arbitrary dimension is time-consuming. The cases that we will be primarily interested in are

(1) the 1-dimensional homology group $H_1(M; \mathbb{Z})$ treated in Section 8.11, and
(2) the 2-dimensional homology group $H_2(M; \mathbb{Z})$ treated in Section 9.3.

In these cases, the homology groups can be characterized more easily without the general machinery of singular simplices and chains. We will therefore follow such an easier approach.

Definition 8.6.1 (Circle with orientation). Let $S^1 \subseteq \mathbb{C}$ be the unit circle, which we think of as a 1-dimensional manifold with an orientation given by a parametrisation in the counterclockwise direction.

Remark 8.6.2. The existence of orientations on parametrized loops in $\mathbb{C}$ is familiar from the course in complex functions. See Sections 8.7 and 8.9 for a more general treatment of orientations and induced orientations.

Definition 8.6.3. A 1-*cycle* C on a manifold M is an integer linear combination

$$C = \sum_i n_i f_i$$

where $n_i \in \mathbb{Z}$ is called the *multiplicity* (ribui), while each

$$f_i : S^1 \to M$$

is a loop given by a smooth map from the circle to M, where each loop f_i is endowed with the orientation coming from the standard circle S^1.

Definition 8.6.4. $Z_1(M; \mathbb{Z})$ is the space of 1-cycles on M.

8.7. Orientation on a manifold

Let M be a connected n-dimensional manifold, so that for every $u \in M$, we have $\dim(T_u^* M) = n$. The n-th exterior algebra at the point u is 1-dimensional, i.e., $\dim\left(\bigwedge^n (T_u^* M)\right) = 1$. Thus, any differential n-form $\eta \in \Omega^n(M)$ can be written in a coordinate chart $A \subseteq M$ as

$$\eta = f(u^1, \ldots, u^n)\, du^1 \wedge \cdots \wedge du^n$$

where f is a function defined in the coordinate patch (A, u).

DEFINITION 8.7.1. A differential n-form $\eta \in \Omega^n(M)$ is *nondegenerate* on M if it is nonvanishing at every point, i.e., $f(u^1, \ldots, u^n) \neq 0$.

REMARK 8.7.2 (Orientability). A nondegenerate n-form may not necessarily exist globally on M. Thus, when $M = \mathbb{RP}^2$ such a form does not exist. Namely, every 2-form on $\mathbb{RP}^2$ necessarily vanishes at some point. This is due to the non-orientability of the real projective plane.

DEFINITION 8.7.3. Two nondegenerate n-forms η and $\tilde{\eta}$ are equivalent if they differ by a positive function $g = g(p) > 0$ on M:

$$\eta \sim \tilde{\eta} \iff (\exists g > 0)(\forall p \in M),\ \eta(p) = g(p)\tilde{\eta}(p). \tag{8.7.1}$$

DEFINITION 8.7.4. An *orientation* on a connected manifold M is an equivalence class in the sense of the equivalence relation (8.7.1) of a nondegenerate n-form $\eta \in \Omega^n(M)$.

DEFINITION 8.7.5. On the circle S^1, the standard orientation is represented by the nondegenerate differential 1-form $d\theta$ (or any positive multiple of it).

REMARK 8.7.6 (Arrow as orientation). The "opposite" orientation on the circle represented by $-d\theta$. This can be thought of as an arrow marked along the circle, indicating the clockwise or counterclockwise direction.

EXAMPLE 8.7.7. The examples:

(1) The area form $dx \wedge dy$ in the plane $\mathbb{R}^2$ or the torus $\mathbb{T}^2$ represents an orientation in the plane or on the torus. The opposite orientation is represented by the form $-dx \wedge dy$.
(2) The Fubini–Study differential 2-form α_{FS} on $\mathbb{CP}^1$ defined by $\alpha_{FS}(u, v) = g(Ju, v)$ (as in Definition 8.1.4) represents an orientation on $\mathbb{CP}^1$. The opposite orientation is represented by the form $-\alpha_{FS}$.

8.8. Interior product $\lrcorner$ of a form by a vector

To define an orientation on cycles in Section 8.9, we will exploit the interior product. Recall that an n-form η at a point $u \in M$ can be thought of as an antisymmetric multilinear form on T_uM. For example, for a 2-form η we can write $\eta(v, w) \in \mathbb{R}$ where $v, w \in T_uM$.

DEFINITION 8.8.1 (Case $n = 2$). Let $v \in T_uM$ be a tangent vector, and $\eta \in \bigwedge_u^2 M$ a 2-form. The *interior product* $\iota_v\eta = v \lrcorner\, \eta \in \bigwedge_u^1 M$ is a 1-form, defined by setting

$$\forall w \in T_uM, \quad (v \lrcorner\, \eta)(w) = \eta(v, w).$$

We define the interior product similarly for n-forms.

DEFINITION 8.8.2. The interior product of an n-form η by a vector v is an $(n-1)$-form denoted $\iota_v(\eta) = v \lrcorner\, \eta$, obtained by substituting v into the first variable of η:

$$\iota_v(\eta) = (v \lrcorner\, \eta)(x_1, \ldots, x_{n-1}) = \eta(v, x_1, \ldots, x_{n-1}).$$

EXAMPLE 8.8.3. Consider the area form $\eta = dx \wedge dy$ in the plane. If $v = \frac{\partial}{\partial x}$ then $v \lrcorner\, \eta = dy$.

LEMMA 8.8.4. *If $w = \frac{\partial}{\partial y}$ then $w \lrcorner\, \eta = -dx$.*

PROOF. The 1-form $w \lrcorner\, \eta$ can be calculated as follows:

$$(\tfrac{\partial}{\partial y} \lrcorner\, \eta)(u) = \eta(\tfrac{\partial}{\partial y}, u) = -\eta(x, \tfrac{\partial}{\partial y}).$$

Thus $(\frac{\partial}{\partial y} \lrcorner\, \eta)(\frac{\partial}{\partial x}) = -1$, $(\frac{\partial}{\partial y} \lrcorner\, \eta)(\frac{\partial}{\partial y}) = 0$. Hence the 1-form $\frac{\partial}{\partial y} \lrcorner\, \eta$ coincides with $-dx$. □

EXAMPLE 8.8.5. In polar coordinates (r, θ), since $\eta = dr \wedge rd\theta$, we have

$$\begin{cases} \frac{\partial}{\partial r} \lrcorner\, \eta = rd\theta \\ \frac{\partial}{\partial \theta} \lrcorner \eta = -rdr \end{cases}$$

REMARK 8.8.6. The interior product satisfies a Leibniz rule:

$$\iota_v(\omega \wedge \eta) = (\iota_v\omega) \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge (\iota_v\eta)$$

(see [**14**, p. 401]). We will not need this rule.[8]

8.9. Induced orientation on the boundary of manifold

We will now exploit the exterior product of Section 8.8 to define the induced orientation on the boundary[9] of a manifold.

DEFINITION 8.9.1. M is an *n-manifold with boundary* if $S \subseteq M$ where S is $(n-1)$-dimensional (not necessarily connected), and an open neighborhood of S in M is diffeomorphic to the cylinder $S \times I$ where $I = (0, 1]$ is a halfclosed interval, where S is identified with $S \times \{1\} \subseteq M$.

[8]In terms of contraction of indices, the interior product can be described as follows. Let v^a be a vector, and η_{bc} a 2-form. Then the contraction is the 1-form $\eta_c = v^a\eta_{ac}$.

[9]Safa with a "sin".

We use the notation ∂M for the boundary of M. Let $(u^1, \ldots, u^{n-1})$ be coordinates in a neighborhood in S, and let $t \in I$ parametrize the interval, so that $-\frac{\partial}{\partial t}$ is an (inward-pointing) vector field transverse (i.e., not tangent) to the boundary $S = S \times \{1\}$.

Definition 8.9.2 (Induced orientation). Let M be an oriented manifold with boundary S, with orientation represented by a nondegenerate form

$$\mathrm{orient}_M \in \Omega^n(M).$$

The induced orientation orient_S on S is generated by the interior product $(-\frac{\partial}{\partial t}) \lrcorner \, \mathrm{orient}_M$ restricted to S, viewed as a form on S itself:

$$\mathrm{orient}_S = (-\tfrac{\partial}{\partial t}) \lrcorner \, \mathrm{orient}_M \in \Omega^{n-1}(S).$$

Remark 8.9.3 (Orientation on connected components). The equivalence class of the $(n-1)$-form $(-\frac{\partial}{\partial t}) \lrcorner \, \mathrm{orient}_M$ on S, relative to the equivalence relation (8.7.1), provides the orientation on each connected component of S.

Example 8.9.4 (area form on disk induces clockwise orientation on boundary circle). Let $\Sigma_0 \subseteq \mathbb{R}^2$ be the unit disk together with the orientation represented by $\eta = dx \wedge dy$. At points other than the origin we can represent η by $\eta = r dr \wedge d\theta$. In particular, such a representation is valid at all points of the circle. The inward-pointing vector field along the boundary circle $S^1 = \partial \Sigma_0$ is the field $-\frac{\partial}{\partial r}$. The induced orientation on S^1 is the 1-form $(-\frac{\partial}{\partial r}) \lrcorner \eta = -r d\theta = -d\theta$ (since $r = 1$ along the circle). This is the clockwise orientation on the circle.

8.10. Surfaces and their boundaries

Definition 8.10.1. We denote by $(\Sigma_g, \partial \Sigma_g)$ a compact orientable surface with boundary $\partial \Sigma_g$, where g is the genus of the surface.

The following is a basic result in the topology of surfaces.

Lemma 8.10.2. *The boundary $\partial \Sigma_g$ is a disjoint union of finitely many circles.*

In Section 8.9 we showed how to induce an orientation on the boundary of an oriented manifold. We used the interior product $\lrcorner$ with the vector field $-\frac{\partial}{\partial t}$ along the boundary as in Definition 8.9.2. We therefore obtain the following corollary.

Corollary 8.10.3. *On an oriented surface $(\Sigma_g, \mathrm{orient}_{\Sigma_g})$, a non-degenerate 2-form $\eta \in \Omega^2(\Sigma_g)$, $\eta \in \mathrm{orient}_{\Sigma_g}$ induces an orientation*

$$(-\tfrac{\partial}{\partial t}) \lrcorner \, \eta$$

on each boundary circle of Σ_g.

COROLLARY 8.10.4. *An orientation on* Σ_g *specifies a choice of an orientation-preserving identification of each connected component of the boundary of* Σ_g *with the standard unit circle* $S^1 \subseteq \mathbb{C}$ *with its counterclockwise orientation as in Definition* 8.7.5.

8.11. Restriction to the boundary

Given a map $f : A \to B$ and a subset $C \subseteq A$, we denote by

$$f\!\restriction_C$$

the restriction of f to C.

DEFINITION 8.11.1. Let M be a manifold (orientable or not). Let $(\Sigma_g, \mathrm{orient}_\Sigma)$ be an oriented surface with boundary. Given a map $h\colon \Sigma_g \to M$, we can form its restriction

$$h\restriction_{\partial\Sigma_g} \tag{8.11.1}$$

to the boundary.

The restriction (8.11.1) is a 1-cycle in $Z_1(M;\mathbb{Z})$ in the sense of Definition 8.6.3, with orientation induced as in Corollary 8.10.3. We define 1-boundaries as the 1-cycles that can be obtained via restriction, as follows.

DEFINITION 8.11.2. A 1-*boundary* in a manifold M is a 1-cycle

$$\sum_i n_i f_i \in Z_1(M;\mathbb{Z})$$

such that there exists a map of an oriented surface $h\colon S \to M$ (not necessarily connected), whose restriction to the boundary satisfies $h\restriction_{\partial S} = \sum_i n_i f_i$.

LEMMA 8.11.3. *The* 1*-boundaries form a group under addition.*

PROOF. The surface S bounding the 1-boundary is not required to be connected. This enables us to take the disjoint union of the bounding surfaces to exhibit a surface bounding the sum. $\square$

DEFINITION 8.11.4. The group of all 1-boundaries in M is denoted

$$B_1(M;\mathbb{Z}) \subseteq Z_1(M;\mathbb{Z}).$$

8.12. 1-homology group

Let M be a differentiable manifold.

Definition 8.12.1. The 1-homology group of M with integer coefficients is the quotient group

$$H_1(M;\mathbb{Z}) = Z_1(M;\mathbb{Z})/B_1(M;\mathbb{Z}).$$

Given a 1-cycle $C \in Z_1(M;\mathbb{Z})$, its homology class is denoted $[C] \in H_1(M;\mathbb{Z})$.

Analogously with de Rham cohomology (see Theorem 7.9.3), we have a pushforward map for homology.

Theorem 8.12.2 (Pushforward homomorphism in homology). *A differentiable map $f\colon M \to N$ induces a natural pushforward homomorphism $f_*\colon H_1(M;\mathbb{Z}) \to H_1(N;\mathbb{Z})$.*

The relation between the fundamental group and the 1-homology group is given by the following theorem. For the notion of abelianisation see Definition 8.4.5.

Theorem 8.12.3. *The 1-homology group $H_1(M;\mathbb{Z})$ is the abelianisation of the fundamental group $\pi_1(M)$:*

$$H_1(M;\mathbb{Z}) = \big(\pi_1(M)\big)^{ab}.$$[10]

The fundamental groups of the real projective plane $\mathbb{RP}^2$ and the 2-torus $\mathbb{T}^2$ are abelian, and therefore isomorphic to their first homology groups, yielding the following examples.

Proposition 8.12.4. *Two cases:*

(1) $H_1(\mathbb{RP}^2;\mathbb{Z}) \simeq \mathbb{Z}_2$,
(2) $H_1(\mathbb{T}^2;\mathbb{Z}) \simeq \mathbb{Z}^2$.

Proposition 8.12.5. *The fundamental group of an orientable closed surface Σ_g of genus g is a group on $2g$ generators with a single relation r which is a product of g commutators:*

$$\pi_1(\Sigma_g) \simeq \big\langle a_1, b_1, \ldots, a_g, b_g \,\big|\, r \big\rangle, \quad r = [a_1,b_1]\cdots[a_g,b_g]. \tag{8.12.1}$$

We obtain the following corollary of Theorem 8.12.3.

Corollary 8.12.6. *Abelianizing the group (8.12.1), we obtain*

$$H_1(\Sigma_g;\mathbb{Z}) \simeq \mathbb{Z}^{2g}\,.$$

[10]Free loops: A significant difference between the fundamental group and the first homology group is the following. While only based loops participate in the definition of the fundamental group, the definition of $H_1(M;\mathbb{Z})$ involves free (not based) loops.

8.13. Length of cycles and of 1-homology classes

Now let M be a Riemannian manifold. We will deal with norms in homology groups $H_k(M;\mathbb{Z})$ associated with Riemannian metrics on M. Assume that an n-dimensional manifold M has a Riemannian metric given in a coordinate patch by the metric coefficients (g_{ij}).

DEFINITION 8.13.1 (volume of loop). Denote by t a parameter on the circle S^1. Given a smooth loop $f\colon S^1 \to M$ with components α^i with respect to a suitable coordinate chart: $f(t) = (\alpha^1(t), \ldots, \alpha^n(t))$. Its volume (i.e., length) with respect to the metric of M is $\mathrm{vol}(f) = \int \sqrt{g_{ij}(f(t))\, \alpha^{i\prime}(t)\alpha^{j\prime}(t)}\; dt$.

As usual, if the loop travels through several charts of M, we work with partitions of M to define the global volume (length) of the loop.

DEFINITION 8.13.2 (Volume of cycle). Consider a 1-cycle $\tilde{C} = \sum_i n_i f_i$ with $n_i \in \mathbb{Z}$ in M. The *volume* (length) of $\tilde{C}$ is the combined length of all the individual loops, with |multiplicity|: $\mathrm{vol}(\tilde{C}) = \sum_i |n_i| \,\mathrm{vol}(f_i)$.

Since we take absolute values of all the integer coefficients, the volume is by definition nonnegative.

DEFINITION 8.13.3 (Volume of homology class). Let $C \in H_1(M;\mathbb{Z})$ be a 1-homology class. We define the volume of C as the infimum of volumes of representative 1-cycles: $\mathrm{vol}(C) = \inf\left\{\mathrm{vol}(\tilde{C})\colon \tilde{C} \in C\right\}$, where the infimum is over all cycles $\tilde{C} = \sum_i n_i f_i$, $n_i \in \mathbb{Z}$, representing the class $C \in H_1(M;\mathbb{Z})$.

EXAMPLE 8.13.4. In the case of the real projective plane, there is only one nontrivial first homology class. The volume of this class with respect to a given metric $\mathbf{g}$ on $\mathbb{RP}^2$ is then the least length of a noncontractible[11] loop for the metric $\mathbf{g}$.[12]

8.14. Multiplicativity of H_1-length on orientable surfaces

For closed orientable surfaces, the volume of 1-homology classes is multiplicative, in the following sense.

[11] lo-kvitza, with kaf

[12] This material is optional. Consider the standard metric g_{can} of constant Gaussian curvature $+1$ on $\mathbb{RP}^2$. Let $C \neq 0$ be the nontrivial class in the group $H_1(\mathbb{RP}^2;\mathbb{Z}) \simeq \mathbb{Z}_2$. Then (1) the volume (length) of C is $\mathrm{vol}(C) = \pi$; (2) the projection to $\mathbb{RP}^2$ of each longitude on the sphere under the double cover $S^2 \to \mathbb{RP}^2$ represents a minimizing closed geodesic in the class C. The standard metric on $\mathbb{RP}^2$ satisfies the boundary case of equality in Pu's inequality; see note preceding Section 9.14.

THEOREM 8.14.1. *Let $M = \Sigma_g$ be a closed orientable surface endowed with a metric. Consider a homology class $C \in H_1(M;\mathbb{Z})$. Then for all $j \in \mathbb{N}$, we have*

$$\mathrm{vol}(jC) = j\,\mathrm{vol}(C), \tag{8.14.1}$$

where jC denotes the class $C + C + \ldots + C$, with j summands.

See note 13.

REMARK 8.14.2. Another way of writing identity (8.14.1) for a homology class C is as follows:

$$\forall j \in \mathbb{N}, \quad \mathrm{vol}(C) = \frac{1}{j}\,\mathrm{vol}(jC). \tag{8.14.2}$$

A similar formula will *define* the stable norm in homology for higher-dimensional manifolds; see formula (9.1.1).[13]

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[13]Remarks toward a proof. We have the following elementary fact about the topology of a *cylinder*.

LEMMA 8.14.3. *A loop γ going around a cylinder twice necessarily has a point of self-intersection enabling a decomposition $\gamma = \gamma_1 \cup \gamma_2$ into a pair of noncontractible loops on the cylinder.*

PROOF. We will give a proof in the case when the loop is the graph of a 4π-periodic function $f(t)$. Consider the difference $h(t) = f(t) - f(t + 2\pi)$. Clearly, $h(t) = -h(t + 2\pi)$. Thus h takes both positive and negative values (if $h(t)$ is positive, then $h(t + 2\pi)$ is negative). By the intermediate function theorem, the function h must have a zero t_0. Then $f(t_0) = f(t_0 + 2\pi)$. Then the parameter value t_0 produces a point of self-intersection of the loop. □

PROOF OF THEOREM 8.14.1. Without loss of generality, we can assume that C is a primitive class (i.e., not a nontrivial integer multiple of another). Choose a representative loop $\tilde{C} \in C$ (for example, a minimizing closed geodesic). Choose the basepoint $p \in M$ to be a point of $\tilde{C}$, and let N be the cover of $M = \Sigma_g$ whose fundamental group is infinite cyclic and is generated by the class of $\tilde{C}$ in $\pi_1(M, p)$. Then N is a cylinder. The lift of $\tilde{C}$ to N represents a generator of $H_1(N;\mathbb{Z})$. To fix ideas, we let $j = 2$. Consider a minimizing loop $\gamma \in 2C$. By Lemma 8.14.3, the loop γ will necessarily intersect itself in a suitable point $p \in N$. This results in a decomposition $\gamma = \gamma_1 \cup \gamma_2$ at the level of loops, with an orientation on each γ_i restricted from γ, where we may assume that $|\gamma_1| \le |\gamma_2|$. We thus obtain a decomposition $[\gamma] = [\gamma_1] + [\gamma_2]$ at the level of homology classes. If one of the γ_i were nullhomologous, the other would give a shorter representative of $2C$ than γ, producing a contradiction. Then $\gamma_1 \in C$ and of length at most half of the length of γ, proving the identity (8.14.1) in the case $j = 2$. The general case follows similarly. □

CHAPTER 9

2-homology groups, stable norm, Gromov's inequality

9.1. Stable norm for higher-dimensional manifolds

In Section 8.14, we defined the volume (length) of 1-homology classes of an orientable surface. The phenomenon of multiplicativity of volume (length) expressed by identity (8.14.2) no longer holds in a higher-dimensional manifold M. Namely, the volume (length) of 1-homology classes in $H_1(M;\mathbb{Z})$ is not necessarily multiplicative. However, the limit as $j \to \infty$ exists and is called the stable norm.

Definition 9.1.1 (Stable norm in homology). Let M be a compact Riemannian manifold. The associated *stable norm* $\| \ \|$ of a class $C \in H_1(M;\mathbb{Z})$ is the limit

$$\|C\| = \lim_{j\to\infty} \frac{1}{j} \operatorname{vol}(jC). \tag{9.1.1}$$

Remark 9.1.2. The existence of the limit is nontrivial and was proved by Federer in [**Fe69**].

Remark 9.1.3. It is obvious from the definition that one has

$$\|C\| \leq \operatorname{vol}(C).$$

However, the inequality may be strict in general. By Theorem 8.14.1, if M is an orientable surface then $\|C\| = \operatorname{vol}(C)$.

Proposition 9.1.4. *The stable norm vanishes for a class of finite order.*

Proof. If $C \in H_1(M;\mathbb{Z})$ is a class of finite order, one has finitely many possibilities for $\operatorname{vol}(jC)$ as j varies. Let V be the maximal such volume. Then the expression $\frac{1}{j}\operatorname{vol}(jC) \leq \frac{V}{j}$ in (9.1.1) tends to zero, proving that $\|C\| = 0$. □

Example 9.1.5. Recall that the fundamental group of the real projective space is the finite cyclic group $\mathbb{Z}_2$. It follows that the stable norm on $H_1(\mathbb{RP}^n;\mathbb{Z}) \simeq \mathbb{Z}_2$ vanishes for $n \geq 2$.

Similarly, if two classes differ by a class of finite order, their stable norms coincide. Thus the stable norm passes to the quotient lattice $L_1(M)$ defined below.

DEFINITION 9.1.6. Let M be a compact manifold. The torsion subgroup of $H_1(M;\mathbb{Z})$ will be denoted $T_1(M) \subseteq H_1(M;\mathbb{Z})$. The quotient lattice $L_1(M)$ is the lattice

$$L_1(M) = H_1(M;\mathbb{Z})/T_1(M).$$

Recall that every finitely generated abelian group is a product $\mathbb{Z}^b \times F$ where F is finite. The Betti numbers were defined in Definition 7.9.2.

PROPOSITION 9.1.7. *Let M be a compact manifold. Then the lattice $L_1(M)$ is isomorphic to $\mathbb{Z}^{b_1(M)}$, where b_1 is the first Betti number of M.*

This is a general result in algebraic topology which we do not pursue here.

9.2. Systole and stable 1-systole

Let M be a non-simply-connected closed manifold endowed with a Riemannian metric $\mathbf{g}$.

DEFINITION 9.2.1. The 1-systole $\operatorname{sys}_1(M,\mathbf{g})$ of a metric $\mathbf{g}$ on a manifold M is the least $\mathbf{g}$-length of a noncontractible loop on M.

Given a metric $\mathbf{g}$, we consider the associated stable norm $\| \ \|$ as in (9.1.1) in the homology group $H_1(M;\mathbb{Z})$. Recall that $\lambda_1(L)$ is the least length of a nonzero vector in a lattice L; see Definition 6.4.1.

DEFINITION 9.2.2. The *stable* 1-*systole* of M, denoted $\operatorname{stsys}_1(M)$, is the least norm of a 1-homology class of infinite order:

$$\begin{aligned}\operatorname{stsys}_1(M) &= \inf\left\{\|C\| \colon C \in H_1(M;\mathbb{Z}) \setminus T_1(M)\right\} \\ &= \lambda_1\left(L_1(M), \| \ \|\right)\end{aligned}$$

COROLLARY 9.2.3. *For an arbitrary metric* $\mathbf{g}$ *on the* 2*-torus* $\mathbb{T}^2$*, the* 1*-systole and the stable* 1*-systole coincide:*

$$\operatorname{sys}_1(\mathbb{T}^2) = \operatorname{stsys}_1(\mathbb{T}^2).$$

PROOF. Since the fundamental group of the torus is abelian, any noncontractible loop represents a nontrivial class in homology, and the corollary is immediate from Theorem 8.14.1.[1] □

[1]For an orientable surface M, the group $H_1(M;\mathbb{Z})$ contains no torsion see Proposition 8.12.5). For M, the inequality of the corollary does not hold as stated

9.3. 2-homology group

Let M be a manifold. We will now define the 2-homology group in a way similar to the procedure for the 1-homology group defined in Section 8.11. An example of particular interest to us is the complex projective space $M = \mathbb{CP}^n$.

Definition 9.3.1. A 2-*cycle* in M is a linear combination $\sum_i n_i f_i$, where $n_i \in \mathbb{Z}$, of maps $f_i \colon \Sigma_{g_i} \to M$ of closed oriented surfaces of arbitrary genus g_i into M.

Remark 9.3.2. A significant difference from the case of 1-homology group where we used a unique source manifold (namely the circle) is that we now allow arbitrary genus for the source surface. Thus, each f_i is a map $f_i \colon \Sigma_{g_i} \to M$ where Σ_{g_i} is a closed surface of genus g_i depending on i.

A 1-boundary was defined in Section 8.6 as the boundary of a map from a surface to M. We define 2-boundaries as follows. Recall that the *orientation* induced on the boundary surface ∂N of an oriented 3-manifold N was defined in Section 8.7 via the interior product $\lrcorner$.

Definition 9.3.3. A 2-*boundary* in M is a 2-cycle $\sum_i n_i f_i$ realizable as the boundary ∂N of a map $F \colon N \to M$ from an oriented 3-manifold N into M:

$$\sum_i n_i f_i = F\downharpoonright_{\partial N},$$

with orientation induced from that of N.

Definition 9.3.4. We let $Z_2(M;\mathbb{Z})$ and $B_2(M;\mathbb{Z})$ be the spaces of 2-cycles and 2-boundaries of M, respectively.

Definition 9.3.5. The 2-homology group of M is the quotient group

$$H_2(M;\mathbb{Z}) = Z_2(M;\mathbb{Z})/B_2(M;\mathbb{Z}).$$

Remark 9.3.6 (Codimension-1 homology). If M is an orientable n-dimensonal manifold, one can define codimension-1 homology group $H_{n-1}(M;\mathbb{Z})$ similarly via $(n-1)$-cycles which are linear combinations of maps from $(n-1)$-dimensional manifolds into M. This will be used in Section 11.4.

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because of possible thin noncontractible hullhomologous necks which are invisible in homology. However, a similar result does hold for the *homology systole* (where unlike the homotopy systole of Definition 9.2.1, the infimum is taken only over homologically nontrivial loops). Namely, the homology systole of M coincides with the stable homology systole of M.

We have a pushforward homomorphism in homology as in Theorem 8.12.2.

THEOREM 9.3.7. *A differentiable map $f\colon M \to N$ between manifolds induces a natural pushforward homomorphism $f_*\colon H_2(M;\mathbb{Z}) \to H_2(N;\mathbb{Z})$.*

THEOREM 9.3.8. *We have the following important cases:*

(1) *For every orientable surface Σ_g of genus $g \geq 0$, $H_2(\Sigma_g;\mathbb{Z}) \simeq \mathbb{Z}$.*
(2) *We have $H_2(S^n;\mathbb{Z}) \simeq \mathbb{Z}$ if $n = 2$ and 0 otherwise.*
(3) *We have $H_2(\mathbb{CP}^n;\mathbb{Z}) \simeq \mathbb{Z}$ for each $n \geq 1$. The group is generated by the class of the 2-cycle given by the inclusion $\mathbb{CP}^1 \to \mathbb{CP}^n$.*

Thus if $f\colon \mathbb{CP}^1 \to \mathbb{CP}^n$ is the inclusion, the induced homomorphism $f_*\colon H_2(\mathbb{CP}^1) \to H_2(\mathbb{CP}^n)$ is an isomorphism.

9.4. Stable norm in 2-homology, 2-systole

Let $(M, \mathbf{g})$ be an n-dimensional Riemannian manifold. Then the stable norm $\|\ \|$ in $H_2(M;\mathbb{Z})$ is defined in a way similar to the case $k = 1$, using the stabilisation formula (9.1.1), as follows. We first define the area of a map of a surface into the manifold M based on the following data.

(1) $U \subseteq \mathbb{R}^2$ is a domain with coordinates (u^1, u^2).
(2) $\underline{x}\colon U \to M$ is smooth (not necessarily a regular parametrisation).
(3) $x_1 = \frac{\partial x}{\partial u^1}$ and $x_2 = \frac{\partial x}{\partial u^2}$ are tangent vectors in T_pM.
(4) $|x_1 \wedge x_2|_g$ is the area of the parallelogram in T_pM spanned by x_1 and x_2 with respect to the metric $\mathbf{g}$ on M.

DEFINITION 9.4.1 (Not the metric coefficients). Let $g_{ij} = \mathbf{g}(x_i, x_j)$ and $\det(g_{ij}) = g_{11}g_{22} - g_{12}^2$. Note that these functions g_{ij} are *not* the "metric coefficients" and may in fact vanish (if for instance $\underline{x}$ is the constant map).

DEFINITION 9.4.2 (Volume of map from a surface). The volume (i.e., area) of the map $\underline{x}\colon U \to M$ is the integral

$$\mathrm{vol}(\underline{x}) = \int_U |x_1 \wedge x_2|_g \, du^1 du^2 = \int_U \sqrt{\det(g_{ij})}\, du^1 du^2.^2 \qquad (9.4.1)$$

[2]The area of the parallelogram is calculated via the usual formula as $|x_1 \wedge x_2| = |x_1|\,|x_2| \sin\alpha$ where α is the angle between the vectors (defined whenever both vectors are nonzero), and $|\ |$ is shorthand for $|\ |_g$. Here $|x_i| = \sqrt{\mathbf{g}(x_i, x_i)}$ where $\mathbf{g}$ is the metric on M.

As in the case of 1-homology, we define the volume of a 2-homology class as follows.

DEFINITION 9.4.3 (Volume of homology class). Let $C \in H_2(M;\mathbb{Z})$ be a homology class. Then $\mathrm{vol}(C)$ is the infimum of volumes of representative 2-cycles $\tilde{C} \in C$.

DEFINITION 9.4.4. The 2-systole $\mathrm{sys}_2(M,g)$ of a Riemannian manifold (M,g) is the least g-volume of a nontrivial 2-homology class of M.

DEFINITION 9.4.5 (Stable norm). In 2-homology, the stable norm of a class $C \in H_2(M;\mathbb{Z})$ is defined by a formula similar to (9.1.1):

$$\|C\| = \lim_{j\to\infty} \frac{1}{j}\,\mathrm{vol}(jC).$$

The stable 2-systole is defined similarly to the case of stable 1-systole as follows.

DEFINITION 9.4.6. The *stable 2-systole* of a Riemannian manifold M is

$$\begin{aligned}\mathrm{stsys}_2(M) &= \inf\left\{\|C\| \colon C \in H_2(M;\mathbb{Z}) \setminus T_2(M)\right\}\\ &= \lambda_1\big(L_2(M), \|\ \|\big),\end{aligned}$$

where $L_2(M)$ is the lattice given by the quotient of $H_2(M;\mathbb{Z})$ by the torsion subgroup $T_2 \subseteq H_2(M;\mathbb{Z})$, and $\|\ \|$ is the stable norm.

The stable 2-systole is the geometric invariant appearing in Gromov's stable systolic inequality for complex projective space (see Section 9.10). This inequality is a higher dimensional analogue of Pu's systolic inequality for the real projective plane $\mathbb{RP}^2$. In note 14 preceding Section 9.14 we review Pu's inequality as motivation for Gromov's inequality (see Section 9.10).[3]

[3] Integration over 1-cycles and over 2-cycles. This material is optional and is intended as motivation for the process of integration of closed forms over cycles in a manifold; see Section 9.5. For example, consider a circle $C_r = \{z \in \mathbb{C} \colon |z| = r\}$ of radius $r > 0$ centered at the origin of $\mathbb{C}$. The counterclockwise orientation turns the circle into a 1-cycle (see Section 8.7). Thus we can think of C_r as an element of the space of cycles in $M = \mathbb{C} \setminus \{0\}$ with integer coefficients: $C_r \in Z_1(M;\mathbb{Z})$. In polar coordinates (r,θ) in M, the closed differential 1-form $d\theta$ is defined everywhere in M. We will denote its restriction to the circle $C_r \subseteq M$ by the same symbol $d\theta$ (Recall that θ is a multi-valued function on M and also on the circle C_r). Note that the 1-cycle $C_r \in Z_1(M)$ satisfies the relation $\int_{C_r} d\theta = 2\pi$, independent of r. Indeed, we can parametrize the circle by the interval $[0,2\pi]$, for instance by means of the parametrisation $re^{i\theta}$, $\theta \in [0,2\pi]$. Thus we can think of θ as a single-valued function on the circle with a point removed: $\theta \colon C_r \setminus \{re^{i0}\} \to \mathbb{R}$, omitting the point $re^{i0} \in \mathbb{R} \subseteq \mathbb{C}$ which corresponds to the values 0

9.5. Duality of homology and de Rham cohomology

In previous chapters, we have developed

(1) a cohomology theory (de Rham cohomology), and
(2) a homology theory (singular homology)

of a manifold M. In this section we will begin to relate the two theories. We start with the following data.

(1) exterior bundle $\bigwedge^k M = \bigwedge^k (T^*M)$ of the manifold M;
(2) $\Omega^k(M)$ on M is the space of sections of the exterior bundle.
(3) a differential k-form $\eta \in \Omega^k(M)$;
(4) $C \in Z_k(M;\mathbb{Z})$ a k-cycle.

Here η can be integrated over C to obtain a real number $\int_C \eta \in \mathbb{R}$.[4] Integration thus defines a pairing, which we will denote by the symbol $\langle\ ,\ \rangle$.[5] We thus obtain the following lemma.

LEMMA 9.5.1. *There exists a pairing between the space $\Omega^k(M)$ of differential k-forms on M, and the space $Z_k(M;\mathbb{Z})$ of k-cycles in M:*

$$\langle\ ,\ \rangle\colon\ \Omega^k(M) \times Z_k(M;\mathbb{Z}) \to \mathbb{R}$$

given by integration.

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or 2π of the parameter θ. (As already mentioned in note 13 in Section 7.12, the closed 1-form $d\theta$, inspite of appearances, is not exact, i.e., it is not a 1-coboundary. This is because θ is not a true function on the circle but only a multi-valued one. Note that the 1-form $rd\theta$ is the volume (length) form of C_r.) We use the parametrisation $re^{i\theta}$, $\theta \in [0, 2\pi]$ to perform the integration using the fundamental theorem of calculus: $\int_{C_r} d\theta = \theta\big]_0^{2\pi} = 2\pi$, proving our claim.

Another phenomenon related to integration of closed forms over cycles is Cauchy's residue theorem: The residue (She'erit in complex function theory) of the function $\frac{1}{z}$ at the origin is 1. Indeed, we have $\oint_{|z|=r} \frac{dz}{z} = \int \frac{ire^{i\theta}d\theta}{re^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$. Hence the residue at the origin of the meromorphic function $\frac{1}{z}$ equals 1 by Cauchy's theorem. Cauchy's theorem on the residues is a special case of the fact that the contour integral depends only on the homology class $[C_r]$ of the loop C_r in the group $H_1(\mathbb{C} \setminus \{0\};\mathbb{Z})$. This in turn is a special case of the general result on integration over cycles in manifolds; see Section 9.5. Similar remarks apply to integrating a 2-form over a 2-cycle in a manifold. Here 2-cycles are taken in the sense of the homology theory developed in Section 9.3. The area of a 2-cycle is calculated by means of formula (9.4.1) from classical differential geometry as in Section 9.4. Namely, the area of a 2-cycle $F(x,y)$ with coordinates $x = u^1$, $y = u^2$, is expressed by a double integral area $= \int_U |F_x \wedge F_y|\, dxdy$, where $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$. From the differential geometric viewpoint, a more correct form of the expression for the area is the double integral $\int_U |F_x \wedge F_y|\, dx \wedge dy$.

[4]As illustrated in footnote 3.

[5]We have defined k-cycles on a manifold M only in the cases $k = 1$ and $k = 2$. These two values are meant throughout this section.

The relation between the two theories is given by the following theorem.

THEOREM 9.5.2 (Stokes generalized). *Let M be a compact manifold. Let $\eta \in Z^k_{\mathrm{dR}}(M)$ be a closed differential form. Let $\tilde{C} \in Z_k(M;\mathbb{Z})$ be a k-cycle. Then the value of $\langle \eta, \tilde{C} \rangle = \int_{\tilde{C}} \eta$ only depends on the following data:*

(1) *the cohomology class $\omega = [\eta] \in H^k_{\mathrm{dR}}(M)$ and*
(2) *the homology class $C = [\tilde{C}] \in H_k(M;\mathbb{Z})$.*

Since the integral vanishes over any torsion class in $H_k(M;\mathbb{Z})$, we obtain the following pairing.

DEFINITION 9.5.3. We have a pairing

$$\langle\ ,\ \rangle\colon \quad H^k_{\mathrm{dR}}(M) \times L_k(M) \to \mathbb{R}, \tag{9.5.1}$$

where $L_k(M) = H_k(M;\mathbb{Z})/T_k(M)$.

Duality between lattices will be understood in the sense of Definition 6.3.4.

DEFINITION 9.5.4 (Integer lattice in cohomology). Using the pairing (9.5.1), we define an integer lattice $L^k_{\mathrm{dR}} = L^k_{\mathrm{dR}}(M)$ in the de Rham cohomology $H^k_{\mathrm{dR}}(M)$ in the following three equivalent ways.

(1) L^k_{dR} is the lattice dual to the homology lattice $L_k(M)$ via the pairing (9.5.1).
(2) $L^k_{\mathrm{dR}} = \left\{\omega \in H^k_{\mathrm{dR}}(M) \colon \int_{\tilde{C}} \eta \in \mathbb{Z} \left(\forall \tilde{C} \in C \in L_k(M), \forall \eta \in \omega\right)\right\}$.
(3) $L^k_{\mathrm{dR}} = \left\{\omega \in H^k_{\mathrm{dR}}(M) \colon \int_{y_i} \omega \in \mathbb{Z} \quad \forall i = 1, \ldots, b\right\}$, for any basis $(y_1, \ldots, y_b)$ for $L_k(M)$.

DEFINITION 9.5.5. An *integer class* in de Rham cohomology $H^k_{\mathrm{dR}}(M)$ is a class contained in the integer lattice $L^k_{\mathrm{dR}}(M)$.

The fundamental result on de Rham cohomology is the following theorem due to de Rham.

THEOREM 9.5.6 (Nondegeneracy of pairing). *Let M be a closed orientable n-dimensional manifold. Then the pairing $\langle\ ,\ \rangle$ between $L_k(M)$ and $L^k_{\mathrm{dR}}(M)$ is non-degenerate.*

COROLLARY 9.5.7. *The lattice $L_k(M)$ is naturally isomorphic to the lattice $\left(L^k_{\mathrm{dR}}(M)\right)^*$, and $L^k_{\mathrm{dR}}(M)$ is naturally isomorphic to the lattice $(L_k(M))^*$.*[6]

[6]The integer lattice in de Rham cohomology can also be thought of as the image, under the de Rham homomorphism $H^k(M;\mathbb{Z}) \to H^k_{\mathrm{dR}}(M)$, of the singular cohomology with integer coefficients. Singular cohomology was not treated in these notes.

There is a version of such duality for *normed* lattices in homology and cohomology, as well; see Corollary 9.8.2.

9.6. Volume form on Riemannian manifolds

Orientations were defined in Section 8.9.

DEFINITION 9.6.1 (Direct basis). Let (M, orient_M) be an oriented manifold of dimension n. Let $u \in M$.[7] A basis for $T^*_u M$ is *direct* if it is compatible with the chosen orientation on M, in the sense that a representative n-form is given by a positive function times the form $du^1 \wedge \cdots \wedge du^n$.

In Section 8.7, we defined the notion of a *nondegenerate* top-dimensional differential form $\zeta \in \Omega^n(M)$.

Now suppose we are given a Riemannian metric on M. Let $\| \, \|_u$ be the corresponding (pointwise) *comass* norm of an alternating (exterior) form as in see Section 6.6.

DEFINITION 9.6.2. A *volume form* on a Riemannian manifold M is a nondegenerate form $\zeta \in \Omega^n(M)$ such that one of the following equivalent conditions is satisfied at every point $u \in M$:

(1) $\|\zeta\|_u = 1$, and with respect to any direct orthonormal basis $\omega_1, \ldots, \omega_n$ for $T^*_u M$, one has $\zeta_u = \omega_1 \wedge \cdots \wedge \omega_n$;
(2) in any coordinate chart $(u^1, \ldots, u^n)$ such that the basis (du^i) is direct, one has $\zeta = \sqrt{\det(g_{ij})}\, du^1 \wedge \ldots \wedge du^n$.

EXAMPLE 9.6.3 (Product form). If η_M is a volume form on M and η_N a volume form on N then the exterior product form $\pi^*\eta_M \wedge \pi^*\eta_N$ is a volume form on $M \times N$, where π^* denotes the pullback to the product by the respective coordinate projection.

DEFINITION 9.6.4. The *volume* of an oriented manifold M is the integral

$$\mathrm{vol}(M) = \int_M \zeta \tag{9.6.1}$$

where ζ is the volume form of M. The volume form is commonly denoted $d\mathrm{vol}_M$.

EXAMPLE 9.6.5. In the case of the unit circle S^1, the 1-form $d\theta$ has unit pointwise norm. Hence $d\theta$ is the volume (length) form of the circle. Therefore $\int_{S^1} d\theta = 2\pi = \mathrm{vol}(S^1)$. The class $[d\theta]$ is not in the integer lattice $L^1_{\mathrm{dR}}(S^1)$, but $[\frac{1}{2\pi} d\theta]$ is; see Lemma 9.6.6.

[7] We use u in place of p so as to avoid the misleading notation $\| \, \|_p$.

LEMMA 9.6.6. *The form*

$$\left[\tfrac{d\theta}{2\pi}\right] \in H^1_{\mathrm{dR}}(S^1) \tag{9.6.2}$$

is a generator of the integer lattice $L^1_{\mathrm{dR}}(S^1) \simeq \mathbb{Z}$.

Indeed, the form (9.6.2) integrates to 1 over S^1.

Let us now consider $\mathbb{CP}^1$ with its standard metric expressible in an affine neighborhood as $\frac{dx^2+dy^2}{(1+x^2+y^2)^2}$. The associated Fubini–Study 2-form is an area form on $\mathbb{CP}^1$.

PROPOSITION 9.6.7. *The Fubini–Study* 2*-form* α_{FS} *divided by* π *represents a generator of the integer lattice in de Rham cohomology:*

$$\left[\tfrac{\alpha_{FS}}{\pi}\right] \in L^2_{\mathrm{dR}}(\mathbb{CP}^1) \simeq \mathbb{Z}\,.$$

PROOF. The Fubini–Study volume 2-form in an affine neighborhood $\mathbb{C} \subseteq \mathbb{CP}^1$ can be written as $\alpha_{FS} = \frac{dx\wedge dy}{(1+x^2+y^2)^2}$. The form integrates to π as shown in Theorem 7.13.1. Therefore $\int_{\mathbb{CP}^1} \frac{\alpha_{FS}}{\pi} = 1$ as required. $\square$

The following elementary lemma will be useful in applications and particularly in the proof of Gromov's stable systolic inequality for complex projective space.

LEMMA 9.6.8. *For each nonnegative function* $h(u) \in C^\infty(M)$ *on a Riemannian manifold* M*, we have a bound*

$$\int_M h(u)\, d\mathrm{vol}_M \le (\max{}_h)\, \mathrm{vol}(M),$$

where $\max_h$ *is the maximal value of* h *on* M*.*

9.7. Comass norms in $\Omega^k(M)$ and in $H^k_{\mathrm{dR}}(M)$

Let M be a Riemannian manifold. Let $u \in M$. We denote by $\| \;\|_u$ the comass norm in the exterior algebra $\bigwedge(T^*_u M)$, as in Definition 6.6.2. We now define a related comass norm on differential forms.

DEFINITION 9.7.1 (Comass on differential forms). The comass norm $\| \;\|_\infty$ of a differential k-form $\eta \in \Omega^k(M)$ is the supremum of the pointwise comass norms:

$$\|\eta\|_\infty = \sup\left\{ \|\eta\|_u \colon u \in M \right\}.$$

Next, we define a notion of comass norm in de Rham cohomology, as follows.

DEFINITION 9.7.2 (Comass on cohomology classes). Let $\omega \in H^k_{\mathrm{dR}}(M)$ be a class in de Rham cohomology. The comass norm $\|\omega\|^*$ is the infimum of comass norms $\|\eta\|_\infty$ of the representative differential k-forms $\eta \in \omega$, or equivalently

$$\|\omega\|^* = \inf_\eta \sup_u \left\{ \|\eta\|_u \colon u \in M, \eta \in \omega \right\},$$

where $\| \|_u$ denotes the pointwise comass norm.

9.8. Duality of comass and stable norms

We start with the following data.

(1) $L_k(M)$ is the lattice defined as the quotient of homology modulo torsion.
(2) $L^k_{\mathrm{dR}}(M)$ is the integer lattice in de Rham cohomology; see Definition 9.5.4.
(3) The two lattices are dual; see Section 9.5.

The following theorem and its corollary assert that the *normed* lattices are dual, as well. Duality of norms was defined in Section 6.1.

THEOREM 9.8.1 (Federer; Gromov). *The stable norm $\| \|$ on $L_k(M)$ and the comass norm $\| \|^*$ on $L^k_{\mathrm{dR}}(M) \subseteq H^k_{\mathrm{dR}}(M)$ are dual to each other.*

For a proof, see Gromov [**8**, Section 4.34, p. 261]; Federer [**Fe74**, Section 4.10, p. 380] using the Hahn–Banach theorem. See also Pansu [**15**, Lemma 17]. Thus we obtain a strengthened version of Corollary 9.5.7.

COROLLARY 9.8.2. *For a Riemannian manifold M, the normed lattices $(L_k(M), \| \|)$ and $(L^k_{\mathrm{dR}}(M), \| \|^*)$ are dual to each other.*

See also note.[8]

[8] Fubini–Study metric on complex projective line. This material is optional as it is a review of Section 7.13, with an eye to generalizing it to complex projective space in Section 9.9. The case of the complex projective line $\mathbb{CP}^1$ is special in that the Gaussian curvature K is constant: $K = 4$. The complex projective line $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} = S^2$ carries a natural metric called the Fubini–Study metric $\mathbf{g}_{FS}$. The Fubini–Study 2-form was discussed in Section 7.6. Explicit formulas for the Fubini–Study metric on $\mathbb{CP}^n$ will be given in Section 9.9. The distance function $d(v, w)$, associated with the Fubini–Study metric, takes its simplest form in homogeneous coordinates $[Z_0, Z_1]$. Given a pair of vectors $v, w \in \mathbb{C}^2$, we have the corresponding complex 1-dimensional subspaces $\mathbb{C}v \subseteq \mathbb{C}^2$ and $\mathbb{C}w \subseteq \mathbb{C}^2$, thought of as points in the projective line. The distance between them will be denoted, briefly, $d([v], [w])$. The distance is defined in terms of the standard Hermitian product $H(v, w)$. Namely,

9.9. Fubini–Study metric on complex projective space

Like $\mathbb{CP}^1$, the complex projective space $\mathbb{CP}^n$ also carries a natural metric called the Fubini–Study metric. The distance function $d([v],[w])$, associated with the Fubini–Study metric, takes its simplest form in homogeneous coordinates $[Z_0,\ldots,Z_n]$ as already mentioned in Theorem 8.1.1. Namely, in homogeneous coordinates $[Z_0,\ldots,Z_n]$, given a pair of vectors $v,w \in \mathbb{C}^{n+1}$, we have $\cos d([v],[w]) = \frac{|H(v,w)|}{|v|\,|w|}$ where $H(v,w) = \left|\sum_{j=0}^n \overline{v_j} w_j\right|$.[9]

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the distance between points $[v],[w]$ in $\mathbb{CP}^1$ is defined by

$$d([v],[w]) = \arccos \frac{|H(v,w)|}{|v|\,|w|}. \tag{9.8.1}$$

For unit vectors, one has $\cos d([v],[w]) = |H(v,w)| = |\overline{v_1}w_1 + \overline{v_2}w_2|$, where the bar stands for complex conjugation. Note that the distance vanishes (i.e. $\cos(d([v],[w])) = 1$) if v and w differ by a complex scalar: $v = \lambda w$. This is consistent with the fact that these vectors define the same point of the complex projective line. In an affine neighborhood $Z_1 \neq 0$ in $\mathbb{CP}^1$, we introduce a coordinate $z = Z_0/Z_1 = x + iy$. Then, with respect to coordinate z, the Fubini–Study metric $\mathbf{g}_{FS}$ on $\mathbb{CP}^1$ corresponding to the distance function (9.8.1) takes the form

$$\mathbf{g}_{FS} = \frac{|dz|^2}{(1+\bar{z}z)^2} = \frac{dx^2+dy^2}{(1+r^2)^2}. \tag{9.8.2}$$

Here $r^2 = \bar{z}z = x^2+y^2$, where r is the first of the usual polar coordinates (r,θ). We recall some elementary global-geometric properties of the metric. The maximal distance between a pair of points of $\mathbb{CP}^1$ with respect to the distance (9.8.1) is $\frac{\pi}{2}$, i.e. its Riemannian diameter is $\frac{\pi}{2}$. Indeed, the arccosine of a positive value does not exceed $\frac{\pi}{2}$. Since the unit sphere metric has Riemannian diameter π, which is twice the value that appears in the theorem, we obtain the following corollary: The complex projective line $\mathbb{CP}^1$ with the Fubini–Study metric g_{FS} is isometric to the sphere of radius $\frac{1}{2}$ in $\mathbb{R}^3$ of total area π and Gaussian curvature $K=4$. Indeed, as calculated in Theorem 7.6.1, the Gaussian curvature K of the Fubini–Study metric $\mathbf{g}_{FS}$ is $K=4$ at every point. The curvature is calculated using the formula $K = -\Delta_{LB}\log f$ where f is the conformal factor $f = \frac{1}{1+r^2}$ as in (9.8.2). Thus we obtain a metric on the 2-sphere $\mathbb{CP}^1$ of constant Gaussian curvature 4. By a general result in Riemannian geometry, such a metric is isometric to a concentric sphere in $\mathbb{R}^3$. The normalisation $K=4$ forces the radius of the sphere to be $\frac{1}{2}$. Hence the total area is π. Alternatively, this results from the Gauss–Bonnet theorem.

[9]Lawson [**13**, p. 50] gives the following formula the the metric in homogeneous coordinates: $ds_0^2 = 4\frac{|Z\Lambda dZ|^2}{|Z|^4}$ where by definition $|Z\Lambda dZ|^2 = |Z|^2|dZ|^2 - |\langle Z, dZ\rangle|^2$.

DEFINITION 9.9.1 (Affine coordinates). In an affine neighborhood of $\mathbb{CP}^n$ defined by the condition $\{Z_0 \neq 0\}$, the *affine coordinates* are

$$z_j = \frac{Z_j}{Z_0} = x_j + iy_j.$$

DEFINITION 9.9.2. With respect to the affine coordinates $(z_1, \ldots, z_n)$, we consider the Hermitian product

$$H_{\text{aff}}(z, w) = \left| \sum_{j=1}^{n} \overline{z_j} w_j \right|.$$

The Fubini–Study metric takes the following form in affine coordinates.[10]

THEOREM 9.9.3. *Let $\mathbb{C}^n \subseteq \mathbb{CP}^n$ be an affine neighborhood. The Fubini–Study metric $g_{FS}(X, X)$ at a point $u \in \mathbb{C}^n$ is given by the formula*[11]

$$g_{FS}(X, X) = \frac{H_{\text{aff}}(X, X)\,(1 + |u|^2) - |H_{\text{aff}}(X, u)|^2}{(1 + |u|^2)^2}. \tag{9.9.1}$$

DEFINITION 9.9.4. The Fubini–Study 2-form α_{FS} on $\mathbb{CP}^n$ is the form $\alpha_{FS}(v, w) = g_{FS}(Jv, w)$ where J is the complex structure.[12]

PROPOSITION 9.9.5. *The form α_{FS} has the following properties.*

(1) *It is a globally defined closed 2-form on $\mathbb{CP}^n$ and has unit comass at every point.*
(2) *Its cohomology class $[\alpha_{FS}]$ spans the line $H^2_{\text{dR}}(\mathbb{CP}^n) \simeq \mathbb{R}$.*
(3) *The class $\frac{1}{\pi}[\alpha_{FS}]$ is a generator of $L^2_{\text{dR}}(\mathbb{CP}^n) \simeq \mathbb{Z}$.*

Gromov's stable systolic inequality[13] concerns metrics on the complex projective space. We will formulate Gromov's stable systolic inequality for complex projective space in Section 9.10. We first review some material from calculus on manifolds.[14]

[10]For general culture we note the following result concerning sectional curvature, which is a generalisation of Gaussian curvature; see [**Car92**]. In the case $n = 1$, we have $|H(X, u)| = |X|\ |u|$ and the formula reduces to (9.8.2). Unlike the case $n = 1$, sectional curvature is not constant when $n \geq 2$: the sectional curvature K of the Fubini–Study metric on $\mathbb{CP}^n$ satisfies $1 \leq K \leq 4$ at every point and in every 2-dimensional direction.

[11]See e.g., `https://en.wikipedia.org/wiki/Fubini-Study_metric`

[12]There is an explicit formula for α_{FS}, as well. Namely α_{FS} in an affine neighborhood is $\alpha_{FS} = i\left(\frac{\sum_j dz_j \wedge \overline{dz}_j}{1+r^2} - \frac{\left(\sum_j \overline{z_j} dz_j\right) \wedge \left(\sum_j z_j \overline{dz}_j\right)}{(1+r^2)^2}\right)$ related to (9.9.1). See [**GriH78**, p. 30] (who divide by 2π).

[13]Gromov has many other systolic inequalities, as well; see [**8**].

[14] Pu's inequality for real projective plane. The material in this subsection is optional. As motivation, we will first consider an analogous inequality for metrics

9.10. Gromov's inequality for complex projective space

DEFINITION 9.10.1 (Fundamental cohomology class). A *fundamental cohomology class* for a closed orientable manifold M of dimension d is a class $\sigma \in H^d_{\mathrm{dR}}(M)$ such that for a d-form $\tilde{\sigma} \in \sigma$, one has

$$\int_M \tilde{\sigma} = 1.$$

EXAMPLE 9.10.2. The 1-form $d\theta$ is the volume form of the unit circle S^1, whereas the class $\frac{1}{2\pi}[d\theta]$ is its fundamental cohomology class.

Recall that the cohomology ring of $\mathbb{CP}^n$ is a truncated polynomial ring (see Theorem 8.3.4). We will use the following related result in the proof of Gromov's inequality.

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on the real projective plane. Recall that the real projective plane $\mathbb{RP}^2$ is a smooth non-orientable surface. Among its elementary properties are the following.

(1) $\pi_1(\mathbb{RP}^2) = H_1(\mathbb{RP}^2; \mathbb{Z}) \simeq \mathbb{Z}_2$.
(2) The orientable double cover of $\mathbb{RP}^2$ is the sphere S^2.
(3) $\mathbb{RP}^2$ is the quotient of S^2 by the action of $\mathbb{Z}_2 = \{e, a\}$, where $a\colon S^2 \to S^2$ is the antipodal map of S^2, namely $a(p) = -p$ when S^2 is thought of as the unit sphere in $\mathbb{R}^3$.
(4) thus we have a smooth map $S^2 \to \mathbb{RP}^2$ which is 2-to-1;
(5) Every metric $\mathbf{g}$ on $\mathbb{RP}^2$ naturally lifts to a $\mathbb{Z}_2$-invariant metric, denoted $\tilde{\mathbf{g}}$, on the double cover S^2.

The systole $\mathrm{sys}_1(\mathbb{RP}^2, \mathbf{g})$ can be thought of in two equivalent ways: (1) the least $\mathbf{g}$-length of a loop representing the nontrivial homology class in $H_1(\mathbb{RP}^2; \mathbb{Z})$; (2) the least $\tilde{\mathbf{g}}$-length of a path joining points p and $a(p)$ on S^2 as p varies over S^2, where $\tilde{g}$ is the metric on S^2 obtained as the lift of $\mathbf{g}$. P. M. Pu's Theorem asserts that every metric $\mathbf{g}$ on the real projective plane $\mathbb{RP}^2$ satisfies the sharp (haduk) inequality $\mathrm{sys}_1(\mathbf{g})^2 \le \frac{\pi}{2}\,\mathrm{area}(\mathbf{g})$, where sys_1 is the 1-systole. The boundary case of equality is attained precisely when $\mathbf{g}$ is of constant Gaussian curvature. Here equality occurs in particular for the standard metric; see note 12. This inequality is proved in Chapter 13 and again in Chapter 14. There is a similarity between Pu's inequality and the isoperimetric inequality for Jordan curves in the plane. Both inequalities relate a length and an area, and both are sharp (optimal), but the inequalitites comparing length and area go in opposite directions. There is an analogous optimal inequality called Loewner inequality for the torus; see Chapter 12. Our main interest in Pu's inequality in this chapter is as motivation for the complex case, where we have an analogous inequality for the stable systole (see Definition 9.4.6): $\mathrm{stsys}_2(\mathbf{g})^n \le n!\,\mathrm{vol}(\mathbf{g})$, for every metric $\mathbf{g}$ on $\mathbb{CP}^n$; see Section 9.10. The Fubini–Study metric satisfies the boundary case of equality. A remark on counterexamples. Is there a systolic analogue for $\mathbb{CP}^2$ of Pu's inequality, of the form $\mathrm{sys}_2(\mathbf{g})^2 \le C\,\mathrm{vol}(\mathbf{g})$, involving the 2-systole rather than the stable 2-systole? It turns out that there are counterexamples to such an inequality due to a discrepancy between the 2-systole and the stable 2-systole; see [**12**].

PROPOSITION 9.10.3. *If ω is a generator of $L^2_{\mathrm{dR}}(\mathbb{CP}^n)$ then the cup power $\omega^{\cup n}$ is a fundamental cohomology class of $\mathbb{CP}^n$.*

We are now ready to state Gromov's stable systolic inequality for complex projective space.

THEOREM 9.10.4 (M. Gromov). *Every Riemannian metric $\mathbf{g}$ on complex projective space $\mathbb{CP}^n$ satisfies the optimal inequality*

$$\operatorname{stsys}_2(\mathbf{g})^n \leq n! \operatorname{vol}_{2n}(\mathbf{g}).$$

The Fubini–Study metric on $\mathbb{CP}^n$ satisfies the boundary case of equality.

The proof of Gromov's inequality is based on the following main ingredients:

(1) the duality between the stable norm and the comass; see Theorem 9.8.1;
(2) Wirtinger inequality given in Theorem 6.10.5, so as to obtain an optimal constant in the inequality.

To clarify the nature of the proof, we will state the following slight generalisation of Gromov's inequality. A similar proof works for the more general theorem.

THEOREM 9.10.5. *Let M be a $2n$-dimensional Riemannian manifold, satisfying the following conditions:*

(1) *$b_2(M) = 1$, i.e., $H^2_{\mathrm{dR}}(M) \simeq \mathbb{R}$;*
(2) *For a nonzero class $\omega \in H^2_{\mathrm{dR}}(M)$, we have $\omega^{\cup n} \neq 0$ in $H^{2n}_{\mathrm{dR}}(M)$.*

Then every Riemannian metric $\mathbf{g}$ on M satisfies the inequality

$$\operatorname{stsys}_2(\mathbf{g})^n \leq n! \operatorname{vol}_{2n}(\mathbf{g}).$$

The proof appears in Sections 9.13 and 9.14.

REMARK 9.10.6. The requirement $b_2 = 1$ will be lifted in Section 10.9. The reason that the cup-product requirement is necessary can be explained as follows: in order for the systole to control the volume, we need to be able to use 2-dimensional cohomology to control the top-dimensional cohomology. See further in Section 9.11.

9.11. Counterexamples to systolic inequalities

We comment on the necessity of the nonvanishing condition for the top cup power of $\omega \in H^2_{\mathrm{dR}}(M)$ in Theorem 9.10.5 in order to ensure the existence of an inequality of type $\operatorname{stsys}_2(M)^n \leq C_n \operatorname{vol}_{2n}(M)$.

PROPOSITION 9.11.1 (Product metrics). *For $n \geq 3$, let M be the $2n$-dimensional manifold given by a product of spheres: $M = S^2 \times S^{2n-2}$. Then for any constant C_n, suitable product metrics $\mathbf{g}$ on M violate the inequality* $\operatorname{stsys}_2(M, \mathbf{g})^n \leq C_n \operatorname{vol}_{2n}(M, \mathbf{g})$.

PROOF. Since $H^2_{\mathrm{dR}}(S^{2n-2}) = 0$, we have $H^2_{\mathrm{dR}}(M) \simeq \mathbb{R}$ from the Künneth formula (see Corollary 7.10.3). For product metrics on M, it is easy to show that the 2-systole and the stable 2-systole coincide, and are equal to the area of the first factor S^2. We then consider the following family of metrics:

(1) we keep the S^2-factor fixed;
(2) we shrink the S^{2n-2}-factor.

To describe the construction in formulas, we will specify the quadratic forms representing the metrics (see Section 3.1 on polarisation). We let $A = S^2$ and $B = S^{2n-2}$. We let g_A be a fixed metric on A (e.g., the metric of a unit 2-sphere), and g_B a fixed metric on B (e.g., the metric of a unit $(2n-2)$-sphere). Consider the family of metrics $\{g_t \colon t > 0\}$ on M defined by

$$g_t = g_A + t^2 g_B. \tag{9.11.1}$$

The effect of multiplying the metric g_B by t^2 is to multiply the length of every tangent vector to B by t. In other words, for the restrictions of the metrics to the two factors we have

$$g_t\lfloor_A = g_A \text{ and } g_t\lfloor_B = t^2 g_B.$$

Then the 2-systole of (M, g_t) as in (9.11.1) remains constant while its volume tends to zero as t tends to zero. This shows that there is no inequality bounding $(\operatorname{stsys}_2(M))^n$ by $\operatorname{vol}(M)$ for any constant. □

9.12. Homology class C and cohomology class ω

DEFINITION 9.12.1. We let

$$C \in H_2(\mathbb{CP}^n; \mathbb{Z})) \simeq \mathbb{Z} \tag{9.12.1}$$

be the generator in homology represented by the inclusion of the 2-sphere $\mathbb{CP}^1 \subseteq \mathbb{CP}^n$, endowed with its natural orientation stemming from the complex structure. Thus, $C = [\mathbb{CP}^1]$.

The following lemma is immediate.

LEMMA 9.12.2. *For any metric g on $\mathbb{CP}^n$, the class C satisfies $\|C\| = \lambda_1(H_2(\mathbb{CP}^n; \mathbb{Z}), \|\ \|)$ where $\|\ \|$ is the stable norm defined by g.*

Let $L^2_{\mathrm{dR}}(\mathbb{CP}^n) \simeq \mathbb{Z}$ be the integer lattice in de Rham cohomology $H^2_{\mathrm{dR}}(\mathbb{CP}^n)$. By Corollary 9.5.7, the normed lattices $\left(L^2_{\mathrm{dR}}, \|\ \|^*\right)$ and $\left(H_2(\mathbb{CP}^n; \mathbb{Z}), \|\ \|\right)$ are dual.

Definition 9.12.3. Let $\omega \in L^2_{\mathrm{dR}}(\mathbb{CP}^n)$ be the generator in cohomology dual to the homology class C, so that $\int_C \omega = 1$.

Here integration is understood in the sense of a representative cycle in the class C and representative 2-form (such as $\frac{1}{\pi}\alpha_{FS}$) in the class ω.

Lemma 9.12.4. *For any metric g on $\mathbb{CP}^n$, the class ω satisfies $\|\omega\|^* = \lambda_1(L^2_{\mathrm{dR}}(\mathbb{CP}^n), \| \, \|^*)$ where $\| \, \|^*$ is the comass norm of g.*

We have the following data.

(1) The class ω is represented by the form $\frac{1}{\pi}\alpha_{FS}$ where α_{FS} is the Fubini–Study form: $\omega = \left[\frac{1}{\pi}\alpha_{FS}\right] \in H^2_{\mathrm{dR}}(\mathbb{CP}^n)$;
(2) The wedge product $\wedge$ in $\Omega^*(M)$ descends to the cup product $\cup$ in $H^*_{\mathrm{dR}}(M) = \oplus_i H^i_{\mathrm{dR}}(M)$ (see Section 8.2);
(3) the cup power $\omega^{\cup n}$ of ω is a fundamental cohomology class for $\mathbb{CP}^n$; see Proposition 9.10.3.

We therefore obtain the following corollary.

Corollary 9.12.5. *Let ω be a generator of the integer lattice $L^2_{\mathrm{dR}}(\mathbb{CP}^n)$ and let $\eta \in \omega$ be a closed differential 2-form. Then*

$$\int_{\mathbb{CP}^n} \eta^{\wedge n} = 1. \tag{9.12.2}$$

We will use the duality of C and ω as well as (9.12.2) to prove Gromov's inequality in Sections 9.13 and 9.14.

9.13. Comass and application of Wirtinger inequality

To prove Gromov's theorem we exploit Wirtinger inequality as follows; cf. [**8**, Theorem 4.36]. Let $\mathbf{g}$ be an arbitrary metric on the $2n$-dimensional manifold M (e.g, on $\mathbb{CP}^n$). The metric defines a Euclidean norm on each tangent and cotangent space, as well as on all the exterior powers. We can then calculate the pointwise comass $\|\eta\|_u$ of η at a point $u \in M$. Recall from Definition 9.7.1 that the the comass $\|\eta\|_\infty$ of a differential k-form $\eta \in \Omega(M)$ is the supremum of the pointwise comass norms of Definition 6.6.2. Recall that in top dimension, the comass and the Euclidean norm at a point coincide.

Theorem 9.13.1. *Let η be a differential 2-form on a $2n$-dimensional Riemannian manifold M. Then the comass of the form $\eta^{\wedge n}$ satisfies the following inequality: $|\eta^{\wedge n}|_\infty \leq n! \, (\|\eta\|_\infty)^n$.*

Proof. By the Wirtinger inequality and Corollary 6.12.4, we obtain the following bound for the norm at u:

$$|\eta^{\wedge n}|_u \leq n! \, (\|\eta\|_u)^n \leq n! \, (\|\eta\|_\infty)^n \,, \tag{9.13.1}$$

where $\| \|_\infty$ is the comass norm on differential forms. Taking the supremum of (9.13.1) as u runs over M, we obtain the result. $\square$

9.14. Proof of Gromov's inequality

In this section, we will complete the proof of Theorem 9.10.5, namely the inequality $\mathrm{stsys}_2(\mathbf{g})^n \leq n! \operatorname{vol}_{2n}(\mathbf{g})$.

Step 1. Let $(M, \mathbf{g})$ be a Riemannian manifold as in the theorem. We consider the following data:

(1) the integer lattice $L^2_{\mathrm{dR}}(M)$ in de Rham cohomology;
(2) the generator $\omega \in L^2_{\mathrm{dR}}(M) \simeq \mathbb{Z}$;
(3) a representative closed differential 2-form $\eta \in \omega$, $\eta \in \Omega^2(M)$;
(4) the volume form $d\operatorname{vol}_M$ of M.

Thus at every point $u \in M$, we have, up to sign, $\eta_u^{\wedge n} = \|\eta^{\wedge n}\|_u \, d\operatorname{vol}_M$. By Proposition 9.10.3, the class $\omega^{\cup n}$ is the fundamental cohomology class of M, represented by the $(2n)$-form $\eta^{\wedge n}$. Thus we have

$$\begin{aligned} 1 &= \left| \int_M \eta^{\wedge n} \right| \\ &\leq \int_M |\eta^{\wedge n}|_u \, d\operatorname{vol}_M \\ &\leq n! \, (\|\eta\|_\infty)^n \operatorname{vol}_{2n}(M, \mathbf{g}). \end{aligned}$$

The last step follows by Wirtinger's inequality as in (9.13.1). We therefore obtain

$$(\forall \eta \in \omega) \quad 1 \leq n! \, (\|\eta\|_\infty)^n \operatorname{vol}_{2n}(M, \mathbf{g}). \tag{9.14.1}$$

This estimate is valid for every 2-form η representing the generator $\omega \in L^2_{\mathrm{dR}}(M) \subseteq H^2_{\mathrm{dR}}(M)$.

Step 2. We take the infimum of the right-hand side in (9.14.1) over all $\eta \in \omega$. This results in the following lower bound for the comass $\|\omega\|^*$ of the cohomology class ω:

$$1 \leq n! \, (\|\omega\|^*)^n \operatorname{vol}_{2n}(M, \mathbf{g}), \tag{9.14.2}$$

where $\| \|^*$ is the comass norm in cohomology. By Lemma 9.12.4, we obtain

$$1 \leq n! \left[\lambda_1\big(L^2_{\mathrm{dR}}(M), \| \|^*\big) \right]^n \operatorname{vol}_{2n}(M, \mathbf{g}). \tag{9.14.3}$$

Step 3. Denote by $\| \|$ the stable norm in homology. Consider the integer lattice $L_2(M) = H_2(M; \mathbb{Z})/T_2$ (in the case of the complex projective space, the torsion subgroup T_2 is trivial and therefore can be left out of the formula). By Lemma 9.12.2, we have $\|C\| = \lambda_1(L_2(M), \| \ \|)$. Multiplying the inequality (9.14.3) by $\lambda_1(L_2(M), \| \ \|)^n$, we obtain

$$\begin{aligned}&\big(\lambda_1(L_2(M), \| \ \|)\big)^n \leq \\ &\qquad n!\Big[\lambda_1\big(L_2(M), \| \ \|\big)\ \lambda_1\big(L^2_{\mathrm{dR}}(M), \| \ \|^*\big)\Big]^n \mathrm{vol}_{2n}(M, \mathbf{g}).\end{aligned} \tag{9.14.4}$$

Since $\mathrm{stsys}_2(\mathbf{g}) = \lambda_1(L_2(M), \| \ \|)$, this is equivalent to

$$\begin{aligned}&\big(\mathrm{stsys}_2(\mathbf{g})\big)^n \leq \\ &\qquad n!\Big[\lambda_1(L_2(M), \| \ \|)\ \lambda_1\left(L^2_{\mathrm{dR}}(M), \| \ \|^*\right)\Big]^n \mathrm{vol}_{2n}(M, \mathbf{g}).\end{aligned} \tag{9.14.5}$$

Step 4. By Theorem 9.8.1, the normed lattices $(L_2(M), \| \ \|)$ and $\big(L^2_{\mathrm{dR}}(M), \| \ \|^*\big)$ are dual to each other. Since $b_2(M) = 1$, we have

$$\lambda_1(L_2(M), \| \ \|)\ \lambda_1\left(L^2_{\mathrm{dR}}, \| \ \|^*\right) = 1 \tag{9.14.6}$$

by Proposition 6.4.2. Therefore formula (9.14.5) implies the inequality $\big(\mathrm{stsys}_2(\mathbf{g})\big)^n \leq n!\,\mathrm{vol}_{2n}(M, \mathbf{g})$, completing the proof of Gromov's inequality.

Step 5. When $M = \mathbb{CP}^n$, equality is attained by the two-point homogeneous Fubini–Study metric. This is because of the following three equalities.

(1) The standard $\mathbb{CP}^1 \subseteq \mathbb{CP}^n$, and all other complex projective lines, are calibrated by the Fubini–Study 2-form α_{FS}, in the sense that $\int_{\mathbb{CP}^1} \alpha_{FS} = \pi$, or $\int_{\mathbb{CP}^1} \frac{\alpha_{FS}}{\pi} = 1$.
(2) $\int_{\mathbb{CP}^n} \left(\frac{\alpha_{FS}}{\pi}\right)^{\wedge n} = 1$.
(3) We have equality for α_{FS} in the Wirtinger inequality at every point.[15]

Remark 9.14.1. A key step in the proof of Gromov's inequality was a metric-independent upper bound as in (9.14.6) for the product $\lambda_1(L)\,\lambda_1(L^*)$ for a pair of dual lattices. Similar upper bounds exist without the assumption $b_1(M) = 1$; see Section 10.2. Such bounds can be used to prove certain generalisations of Gromov's stable systolic inequality; see Section 10.9.

[15]In the special case $n = 2$ we obtain the following inequality analogous to Pu's inequality of Section 14. Every metric $\mathbf{g}$ on the complex projective plane satisfies the optimal inequality

$$\mathrm{stsys}_2(\mathbb{CP}^2, \mathbf{g})^2 \leq 2\,\mathrm{vol}_4(\mathbb{CP}^2, \mathbf{g}). \tag{9.14.7}$$

CHAPTER 10

Generalizing Gromov's inequality

10.1. Inequality for quaternionic projective plane

For the quaternionic projective plane $\mathbb{H}P^2$, the analysis of the constant in the stable systolic inequality involves an analysis of 4-forms on $\mathbb{R}^8$. Here $\mathbb{H}P^1$ is S^4, $\dim(\mathbb{H}P^2) = 8$, and the inclusion $\mathbb{H}P^1 \subseteq \mathbb{H}P^2$ is a 4-cycle representing a generator of $H_4(\mathbb{H}P^2;\mathbb{Z}) \simeq \mathbb{Z}$.

THEOREM 10.1.1. *The quaternionic projective plane* $\mathbb{H}P^2$ *satisfies the inequality* $(\mathrm{stsys}_4)^2 \leq 14\,\mathrm{vol}_8$.

However, the optimal constant is unknown. It is only known to be in the interval $[6, 14]$ (see [**Bangert et al.**, Proposition 1.4]). The symmetric metric is not optimal in this case, and has a systolic ratio of only $\frac{10}{3}$.

10.2. Successive minima

To develop generalisations of Gromov's inequality to other Betti numbers, we need the notion of successive minima of a lattice.

DEFINITION 10.2.1. Consider a lattice L in a Banach space $(B, \|\ \|)$ of dimension b. Let $1 \leq k \leq b$. The *k-th successive minimum* of L, denoted $\lambda_k(L) = \lambda_k(L, \|\ \|)$, is the least number λ such that there exists a linearly independent k-tuple $(x_1, \ldots, x_k)$ of elements in L satisfying $\|x_i\| \leq \lambda$ for all $i = 1, \ldots, k$.

To obtain stable systolic inequalities for manifolds M generalizing Gromov's inequality for $\mathbb{CP}^n$, we will exploit upper bounds for the product $\lambda_1(L)\,\lambda_b(L^*)$ where $b = b_2(M)$ is the second Betti number of M, whereas $L = H_2(M;\mathbb{Z})/T_2$, and $\lambda_b(L^*)$ is the b-th successive minimum of $L^* = L^2_{\mathrm{dR}}(M)$. Such upper bounds can be used to prove more general stable systolic inequalities (though the constant obtained is rarely sharp), modulo an appropriate condition on the cohomology of M, namely that its fundamental cohomology class is a cup product of 2–dimensional classes; see e.g., [**BK03**]. We will use the following basic fact from algebraic topology. Recall the following from Corollary 7.10.3 (immediate from the Künneth formula (7.10.1)).

Theorem 10.2.2. *Let M and N be connected manifolds. If either M or N is simply connected then*

$$H^2_{\mathrm{dR}}(M \times N) = H^2_{\mathrm{dR}}(M) + H^2_{\mathrm{dR}}(N).$$

10.3. First attempt toward systolic inequality on $S^2 \times S^2$

In Section 10.10, we will present a general result relating stsys_2 and the volume of a $(2n)$-dimensional manifold. To introduce the techniques used in the proof, we first consider the case of the product of two spheres.

Let $M = S^2 \times S^2$. Then $H^2_{\mathrm{dR}}(M) \simeq \mathbb{R}^2$ by Theorem 10.2.2. Let

$$\omega_1, \omega_2 \in L^2_{\mathrm{dR}}(M) \tag{10.3.1}$$

be the generators corresponding to each of the factors S^2. The ω_i are represented by the pullback 2-forms (see Definition 5.6.2) by the two coordinate projections $\pi_{S^2} : M \to S^2$. Here

(1) each ω_i is represented by the normalized area form of the 2-sphere;
(2) the cup-product class $\omega_1 \cup \omega_2$ is the fundamental cohomology class of M.

Proposition 10.3.1. *Let $M = S^2 \times S^2$, and consider the class $\omega = \omega_1 + \omega_2 \in L^2_{\mathrm{dR}}(M)$. Let g be a metric on M. Let $\| \, \|^*$ be the associated comass norm in 2-cohomology. Then $1 \le \|\omega\|^* \sqrt{\mathrm{vol}(M,g)}$.*[1]

Proof. Since 2-forms commute, we have

$$w^{\cup 2} = 2\omega_1^{\cup 2} + 2\omega_1 \cup \omega_2 + \omega_2^{\cup 2} = 2\omega_1 \cup \omega_2. \tag{10.3.2}$$

Therefore if $\sigma \in \omega$ is any representative 2-form, then $\int_M \sigma \wedge \sigma \ge 2$. Applying Wirtinger's inequality as in Chapter 9, we obtain

$$2 \le 2! \int_M \|\sigma\|_u^2 \, d\,\mathrm{vol}_M \le \|\sigma\|_\infty^2 \,\mathrm{vol}(M).$$

Minimizing over all representative 2-forms $\sigma \in \omega$, we obtain

$$1 \le (\|\omega\|^*)^2 \,\mathrm{vol}(M), \tag{10.3.3}$$

as required. $\square$

Inequality (10.3.3) will provide a stable systolic inequality once we can control the comass of ω or another suitable class in $L^2_{\mathrm{dR}}(M)$.

[1] What prevents us from replacing S^2 by S^n in this section is that there is no ready-made analog of Wirtinger for powers of an n-form. For example, for S^4 the constant seems to be 14 instead of 2 due to Spin(7)-holonomy, etc.

REMARK 10.3.2. At this stage, we lack control over the comass of the class $\omega \in L^2_{\mathrm{dR}}(M)$. One fixes this up by choosing, instead, a class from a "short" linearly independent set. The argument will be continued in Section 10.7.

10.4. Standard fundamental domain

Some of the material in this section already appeared in Section 3.4.7.

DEFINITION 10.4.1. The *standard fundamental domain*[2] $D_0 \subseteq \mathbb{C}$ is the domain

$$D_0 = \left\{ z \in \mathbb{C}\colon \ |z| \geq 1, \ -\tfrac{1}{2} \leq \mathrm{Re}(z) < \tfrac{1}{2}, \ \mathrm{Im}(z) > 0 \right\} \tag{10.4.1}$$

DEFINITION 10.4.2. Two lattices in $\mathbb{C}$ are *similar* if they differ by a multiplicative complex scalar.

LEMMA 10.4.3. *Given a lattice $\tilde{L} \subseteq \mathbb{C}$, there is a similar lattice $L \subseteq \mathbb{C}$ and a $\mathbb{Z}$-basis $(1, \tau)$ for L such that $\tau \in D_0$.*

PROOF. Consider a lattice $\tilde{L} \subseteq \mathbb{C}$.

Step 1. Choose a "shortest" nonzero element $z \in \tilde{L}$, *i.e.* we have $|z| = \lambda_1(\tilde{L})$. We replace $\tilde{L}$ by the similar lattice $L = z^{-1}\tilde{L}$. Then $\lambda_1(L) = 1$ and the complex number $+1 \in \mathbb{C}$ is a shortest element in the new lattice L.

Step 2. We complete the element $+1 \in L$ to a $\mathbb{Z}$-basis

$$(1, \tau') \tag{10.4.2}$$

for L. By replacing τ' by $-\tau'$ if necessary, we can ensure the condition of positive imaginary part.

Step 3. If the real part $\mathrm{Re}(\tau')$ of τ' does not satisfy the condition defining the domain D_0, we adjust τ' by adding to it a suitable integer, i.e., replacing it by $\tau' + n$, so that the result $\tau \in L$ satisfies the condition $-\frac{1}{2} \leq \mathrm{Re}(\tau) < \frac{1}{2}$. Here $\mathrm{Re}(\tau)$ can be expressed in terms of the fractional part function $\{\cdot\}$ as $\left\{\mathrm{Re}(\tau') + \frac{1}{2}\right\} - \frac{1}{2}$, so that $\tau = \left\{\mathrm{Re}(\tau') + \frac{1}{2}\right\} - \frac{1}{2} + i\,\mathrm{Im}(\tau')$.

Step 4. Since $\tau \in L$, we have $|\tau| \geq \lambda_1(L) = 1$. Hence τ lies in the domain D_0 of Definition 10.4.1. $\square$

DEFINITION 10.4.4. The number $\tau(L) \in D_0$ is called the *conformal parameter* of the lattice L or any similar lattice.

[2]It is a fundamental domain for the action of the group $\mathrm{PSL}(2, \mathbb{Z})$ in the upperhalf plane of $\mathbb{C}$.

We have the following immediate consequence of the geometry of the standard fundamental domain.

Corollary 10.4.5. *Each lattice $L \subseteq \mathbb{C}$ has a basis (u, v) such that the angle α between u and v is between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, so that $\sin\alpha \geq \frac{\sqrt{3}}{2}$.*

10.5. Lattices in $\mathbb{C}$

The number $\frac{2}{\sqrt{3}}$ will occur a number of times throughout this section. To shorten the formulas, we introduce the notation $\gamma = \frac{2}{\sqrt{3}}$.

Proposition 10.5.1. *Let $L \subseteq \mathbb{C}$ be a lattice and $L^* \subseteq \mathbb{C}$ its dual lattice. Then $\lambda_1(L)\,\lambda_1(L^*) \leq \gamma$.*

Proof. For each pair of dual lattices, the covolumes (coareas in our case) multiply to 1 by Corollary 6.3.7. We can therefore scale our lattices L and L^* so that both would be of unit coarea. Let (u, v) be a basis for L as in Corollary 10.4.5. To fix ideas, assume that $|u| \leq |v|$. Then $\operatorname{area}(\mathbb{C}/L) = |u|\,|v| \sin\alpha = 1$ where α is the angle between them. By Corollary 10.4.5, $\sin\alpha \geq \frac{1}{\gamma}$. It follows that $|u|\,|v| \leq \gamma$ and therefore $|u| \leq \sqrt{\gamma}$. Similarly, we find an element $w \in L^*$ of length at most $\sqrt{\gamma}$. Therefore $|u|\,|w| \leq \gamma$, as required.[3] $\square$

In Section 10.7, we will need the following stronger bound. Let $\lambda_2(L)$ be the second successive minimum of a lattice L, as in Section 10.2.

Proposition 10.5.2. *Let $L \subseteq \mathbb{C}$ be a lattice and $L^* \subseteq \mathbb{C}$ its dual lattice. Then $\lambda_1(L)\,\lambda_2(L^*) \leq \gamma$.*

Proof. Since the product $\lambda_1(L)\,\lambda_2(L^*)$ is scale-invariant, we can normalize our lattice L so that $(1, \tau)$ is a basis for L with $\tau \in D_0$. Then

$$\begin{cases} \lambda_1(L) = 1 \\ \lambda_2(L) = |\tau|. \end{cases}$$

Let α be the angle between 1 and τ, i.e., $\tau = re^{i\alpha}$ where $r \geq 1$. The corresponding matrix is

$$\begin{pmatrix} 1 & \operatorname{Re}(\tau) \\ 0 & \operatorname{Im}(\tau) \end{pmatrix}$$

[3] We have the following stronger version. We say that a basis (u, v) for L is *optimal* if (u, v) is similar to $(1, \tau)$ where $\tau \in D_0$. Let $L \subseteq \mathbb{C}$ be a lattice and L^* its dual lattice. Let α be the angle between elements of an optimal basis for L, so that $\alpha \in [\frac{\pi}{3}, \frac{2\pi}{3}]$. Then $\lambda_1(L)\lambda_1(L^*) \leq \frac{1}{\sin\alpha}$. Indeed, we normalize both lattices to unit covolume. Next, we choose $u \in L$ so that $|u| \leq \left(\frac{1}{\sin\alpha}\right)^{1/2}$, and similarly for L^*, using the fact that its optimal angle is $\pi - \alpha$.

and its inverse matrix is

$$\frac{1}{\mathrm{Im}(\tau)}\begin{pmatrix}\mathrm{Im}(\tau) & -\mathrm{Re}(\tau)\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -\cot\alpha\\ 0 & \mathrm{Im}(\tau)^{-1}\end{pmatrix}$$

so that $\lambda_2(L^*) = \sqrt{1+\cot^2\alpha} = \frac{1}{\sin\alpha}$ by Proposition 6.3.2. By Corollary 10.4.5, $\sin\alpha \geq \gamma^{-1}$. Hence $\lambda_1(L)\,\lambda_2(L^*) = \frac{1}{\sin\alpha} \leq \gamma$, as required. □

10.6. Application of John's theorem

Proposition 10.6.1. *Let $(B, \|\ \|)$ be a 2-dimensional Banach space. Let $L \subseteq B$ be a lattice. Then $\lambda_1(L, \|\ \|)\,\lambda_2(L^*, \|\ \|^*) \leq \sqrt{\frac{8}{3}} < 2$.*

Proof. By F. John's theorem, there is a pair of Euclidean norms on B such that the ratio of the two norms is at most $\sqrt{2}$. In other words, for all $u \in B$, we have $|u| \leq \|u\| \leq \sqrt{2}\,|u|$. For the dual norm $\|\ \|^*$ the inequalities go in the opposite direction:

$$\frac{1}{\sqrt{2}}\,|u| \leq \|u\|^* \leq |u|$$

(here we identify the Eulidean norm and its dual norm). Applying Proposition 10.5.2, we obtain $\lambda_1\big(L, |\ |\big)\,\lambda_2\big(L^*, |\ |\big) \leq \frac{2}{\sqrt{3}}$. Combining this with the above, we obtain

$$\lambda_1(L, \|\ \|)\,\lambda_2(L^*, \|\ \|^*) \leq \sqrt{2}\,\lambda_1(L, |\ |)\,\lambda_2(L^*, |\ |) \leq \tfrac{2\sqrt{2}}{\sqrt{3}},$$

as required. □

10.7. Proof of systolic inequality for $S^2 \times S^2$

To complete the argument of Section 10.3 for $S^2 \times S^2$, we will use the bound of Section 10.6.

Theorem 10.7.1. *The manifold $M = S^2 \times S^2$ with an arbitrary metric g satisfies the inequality*

$$\mathrm{stsys}_2(M, g) \leq 4\sqrt{\mathrm{vol}(M, g)}. \tag{10.7.1}$$

Proof. Let $\|\ \|^*$ be the comass norm in $H^2_{\mathrm{dR}}(M)$. Choose a linearly independent pair

$$u, v \in L^2_{\mathrm{dR}}(M)$$

each of which has comass at most $\lambda_2\big(L^2_{\mathrm{dR}}(M), \|\ \|^*\big)$, where λ_2 is the second successive minimum. We now consider two cases.

Case 1. Suppose one of the pair, say u, has nonzero cup square in $H^4_{\mathrm{dR}}(M)$, so that

$$\left| \int_M u^{\cup 2} \right| \geq 2,^{4}$$

where we used $u^{\cup 2}$ as shorthand for $\eta^{\wedge 2}$ where $\eta \in u$ is a representative 2-form. We then use the class u instead of ω in (10.3.3). Continuing the calculation, we obtain (by minimizing the comass norm as η runs over u)

$$2 \leq 2!\,(\|u\|^*)^2 \operatorname{vol}(M).$$

Our choice of u therefore implies

$$1 \leq \left[\lambda_2\big(L^2_{\mathrm{dR}}(M), \|\ \|^*\big)\right]^2 \operatorname{vol}(M)$$

or equivalently

$$1 \leq \lambda_2\big(L^2_{\mathrm{dR}}(M), \|\ \|^*\big)\sqrt{\operatorname{vol}(M)}.$$

Multiplying both sides by $\lambda_1(L_2(M), \|\ \|)$ gives

$$\lambda_1(L_2(M), \|\ \|) \leq \lambda_1\big(L_2(M), \|\ \|\big)\ \ \lambda_2\left(L^2_{\mathrm{dR}}(M), \|\ \|^*\right)\sqrt{\operatorname{vol}(M)}.$$

Since by definition $\operatorname{stsys}_2(M) = \lambda_1(L_2(M), \|\ \|)$, we obtain

$$\operatorname{stsys}_2(M) \leq \lambda_1\big(L_2(M), \|\ \|\big)\ \ \lambda_2\left(L^2_{\mathrm{dR}}(M), \|\ \|^*\right)\sqrt{\operatorname{vol}(M)}.$$

Then Proposition 10.6.1 yields

$$\operatorname{stsys}_2(M) \leq 2\sqrt{\operatorname{vol}(M)},$$

implying the bound (10.7.1).

Case 2. In the remaining case, we have both $u^{\cup 2} = 0$ and $v^{\cup 2} = 0$ in $H^4_{\mathrm{dR}}(M)$. Consider again the generators ω_1 and ω_2 in $H^2_{\mathrm{dR}}(M)$. By linearity,

$$a\omega_1 \cup b\omega_2 = 2ab\,\omega_1 \cup \omega_2.$$

Therefore a class with vanishing cup-square in $S^2 \times S^2$ is necessarily proportional to one of the generators ω_1, ω_2 from (10.3.1). Thus u and v are proportional to ω_1 and ω_2. Therefore the classes ω_1, ω_2 themselves have norm at most that of the classes u, v. We can therefore continue the argument in this case with the pair ω_1, ω_2 in place of u, v, namely assume that $\|\omega_i\|^* \leq \lambda_2(L^2_{\mathrm{dR}}(M), \|\ \|^*)$. By (10.3.2), the class $\omega = \omega_1 + \omega_2$ has nonzero cup-square. Its comass satisfies

$$\|\omega\|^* \leq \|\omega_1\|^* + \|\omega_2\|^* \leq 2\,\lambda_2\left(L^2_{\mathrm{dR}}(M), \|\ \|^*\right),$$

[4]The intersection form being even for $S^2 \times S^2$, any square will be an even class.

with an extra factor of 2. Arguing as in Case 1 with ω in place of u, we obtain the estimate

$$\mathrm{stsys}_2(M) \leq 4\sqrt{\mathrm{vol}(M)}$$

as required. □

10.8. Counterexample to systolic inequality on $S^2 \times S^2$

Systolic inequalities for (unstable) systoles were conjectured by Marcel Berger starting in the early 1970s. It came somewhat as a surprise in the 1990s that when k-systoles for $k \geq 2$ are involved, counterexamples typically exist.

THEOREM 10.8.1. *The ratio* $\frac{\mathrm{sys}_2}{\sqrt{\mathrm{vol}_4}}$ *can be made arbitrarily large for suitable metrics on* $S^2 \times S^2$.

Such metrics are constructed in [**12**]. For related constructions on $S^1 \times S^3$ see Section 11.4.

10.9. Optimal stable systolic inequality for 4-manifolds

Let $k \geq 1$. The successive minimum λ_k of a normed lattice $(L, \| \ \|)$ was defined in Section 10.2 as the least number λ such that there exists a linearly independent k-tuple $(u_1, \ldots, u_k)$ of elements in L satisfying $\|u_i\| \leq \lambda$ for all $i = 1, \ldots, k$. The (ordinary) Hermite constant is discussed in Section 12.1 below.

DEFINITION 10.9.1. The *generalized Hermite constant* $\Gamma_b > 0$ is the supremum of $\lambda_1(L)\,\lambda_b(L^*)$ over all lattices L in all b-dimensional Banach spaces, where L^* is the lattice dual to L.

EXAMPLE 10.9.2. By Proposition 10.6.1, we have $\Gamma_2 \leq 2$.

We will need the Hodge star operator.

DEFINITION 10.9.3. Let (e_i) be an orthonormal basis for $\mathbb{R}^4$. Let $\gamma = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ be its volume form. Consider the basis of 2-forms given by $e_i \wedge e_j$ where $i < j$. The Hodge star operator $*\colon \bigwedge^2(\mathbb{R}^4) \to \bigwedge^2(\mathbb{R}^4)$ is defined as $*(e_1 \wedge e_2) = e_3 \wedge e_4$ and in general

$$*(e_i \wedge e_j) = \mathrm{sign}_{ij}\, e_k \wedge e_\ell$$

where $\{k, \ell\}$ is the complementary set to $\{i, j\}$ and the sign is chosen in such a way that $e_i \wedge e_j \wedge *(e_i \wedge e_j) = \gamma$.

THEOREM 10.9.4. *Every compact orientable* 4*-manifold with* $b = b_2(M) > 0$ *satisfies the systolic inequality*

$$\mathrm{stsys}_2(M)^2 \leq 2!\,(\Gamma_b)^2\,\mathrm{vol}(M).$$

Proof. If $b_2(M) \neq 0$ then by Poincaré duality, there is a product $\alpha \cup \beta$ of 2-dimensional classes α and β which is nonzero in $H^4_{\mathrm{dR}}(M)$. Decomposing both α and β with respect to a short spanning set for $L^2_{\mathrm{dR}}(M)$, we can assume that each has comass at most $\lambda_b(L^2_{\mathrm{dR}}(M), \| \|^*)$.

In $\mathbb{R}^4$, if we have a wedge $b \wedge c$ of a pair of 2-forms b, c, then $|b \wedge c| = |\langle b, *c\rangle|$ where $*$ is the Hodge star. Now $|\langle b, *c\rangle| \le |b|\,|c|$ by Cauchy-Schwarz. Each of the Euclidean norms of b and c at most $\sqrt{2}$ times the corresponding comass in $\mathbb{R}^4$. Hence $|b \wedge c| \le 2\,\|b\|^*\,\|c\|^*$. Since $\alpha \cup \beta \neq 0$ it follows that $1 \le 2\,\|\alpha\|^* \|\beta\|^* \operatorname{vol}(M)$. Hence

$$(\mathrm{stsys}_2)^2 \le 2\,\lambda_1(L_2(M;\mathbb{Z}))^2 \|\alpha\|^* \|\beta\|^* \operatorname{vol}(M) \le 2\,(\Gamma_b)^2 \operatorname{vol}(M),$$

as required. □

When $b = 1$ we have $\Gamma_b = 1$ so the inequality reduces to Gromov's inequality in that case.

Example 10.9.5. The torus T^4 has $b_2(T^2) = \binom{4}{2} = 6$. Therefore every metric on T^4 satisfies the bound $\frac{\mathrm{stsys}_2^2}{\mathrm{vol}} \le 2\,(\Gamma_6)^2$.

10.10. Stable systolic inequality in dimension $2n$ for any b_2

In this section, we study a relation between the geometry and the topology of a manifold. It turns out that a certain topological condition suffices to guarantee the existence of a geometric inequality, as in Theorem 10.10.1.

Combining John's theorem with classical upper bounds in the Euclidean case, one can prove effective upper bounds for the constant Γ_b. Since the inequalities we will be able to prove are not optimal, we will not concern ourselves with estimating the constant.

Theorem 10.10.1. *Let M be an orientable manifold of dimension $2n$. Assume that its fundamental cohomology class is expressible as a cup product of classes from $H^2_{\mathrm{dR}}(M)$. Then all metrics g on M satisfy a stable systolic inequality*

$$\mathrm{stsys}_2(M, g) \le C_n\,\Gamma_b \sqrt[n]{\operatorname{vol}(M, g)}$$

for a suitable constant $C_n > 0$ depending only on the dimension of M.

Proof. The idea is to exploit a decomposition with respect to a short basis for $L^2_{\mathrm{dR}}(M)$ as in Section 10.3.

Step 1. In the integer lattice $L^2_{\mathrm{dR}}(M)$ in de Rham cohomology, choose a linearly independent set $(u_1, \ldots, u_b)$ where each u_j is of comass at most $\lambda_b\big(L^2_{\mathrm{dR}}(M), \|\ \|^*\big)$, where $b = b_2(M)$. By the assumption of the theorem, there exists a nonzero cup-product

$$\alpha_1 \cup \cdots \cup \alpha_n \neq 0 \in H^{2n}_{\mathrm{dR}}(M) \tag{10.10.1}$$

where $\alpha_i \in H^2_{\mathrm{dR}}(M)$ for all i. Each class α_i is a linear combination of the integer classes $(u_1, \ldots, u_b)$ with real coefficients:

$$\alpha_1 = \sum_{j_1=1}^{b} a_{1,j_1} u_{j_1}, \ \ldots, \ \alpha_n = \sum_{j_n=1}^{b} a_{n,j_n} u_{j_n},$$

where each of the indices $j_1, \ldots, j_n$ runs from 1 to $b = b_2(M)$. Then assumption (10.10.1) becomes

$$\Big(\sum_{j_1} a_{1,j_1} u_{j_1} \cup \cdots \cup \sum_{j_n} a_{n,j_n} u_{j_n}\Big) \neq 0 \in H^{2n}_{\mathrm{dR}}(M).$$

By linearity,

$$\sum_{j_1} a_{1,j_1} \cdots \sum_{j_n} a_{n,j_n} \left(u_{j_1} \cup \cdots \cup u_{j_n}\right) \neq 0 \in H^{2n}_{\mathrm{dR}}(M). \tag{10.10.2}$$

It follows that one of the summands in (10.10.2) must be nonzero. Hence $u_{j_1} \cup \cdots \cup u_{j_n} \neq 0$ for suitable indices.

Step 2. By Step 1, there exists a suitable cup product

$$u_{j_1} \cup \cdots \cup u_{j_n} \neq 0 \tag{10.10.3}$$

which is a nonzero class in $L^{2n}_{\mathrm{dR}}(M)$. Note that the classes u_j may occur in the product (10.10.3) with repetitions (as they do when M is the complex projective space). By our choice of the integer classes u_j, all of the 2-dimensional classes occurring in the product (10.10.3) have comass at most $\|u_j\|^* \leq \lambda_b\big(L^2_{\mathrm{dR}}(M), \|\ \|^*\big)$.

Step 3. Integrating the representing 2-forms as in Section 10.2, we obtain

$$1 \leq C_n \Big[\lambda_b\big(L^2_{\mathrm{dR}}(M), \|\ \|^*\big)\Big]^n \operatorname{vol}(M),$$

where we don't keep track of the precise constant C_n because the resuling systolic inequality will not be optimal anyway. Equivalently, we have

$$1 \leq C_n\, \lambda_b\big(L^2_{\mathrm{dR}}(M), \|\ \|^*\big) \sqrt[n]{\operatorname{vol}(M)} \tag{10.10.4}$$

(for a different constant depending only on n). We now multiply inequality (10.10.4) on both sides by $\lambda_1(L_2(M), \|\ \|)$ to obtain

$$\lambda_1(L_2(M), \|\ \|) \leq C_n\, \lambda_1(L_2(M), \|\ \|)\ \ \lambda_b\left(L^2_{\mathrm{dR}}(M), \|\ \|^*\right) \sqrt[n]{\operatorname{vol}(M)}.$$

By definition of the stable systole and using Definition 10.9.1 of the generalized Hermite constant, we obtain

$$\begin{aligned}\operatorname{stsys}_2(M) &\le C_n\,\lambda_1(L_2(M),\|\ \|)\ \lambda_b\left(L^2_{\mathrm{dR}}(M),\|\ \|^*\right)\ \sqrt[n]{\operatorname{vol}(M)}\\ &\le C_n\,\Gamma_b\,\sqrt[n]{\operatorname{vol}(M)}\end{aligned}$$

as required. □

Conjecture 10.10.2. *Let M be an orientable manifold of dimension $2n$. Assume that its fundamental cohomology class is expressible as a cup product of classes from $H^2_{\mathrm{dR}}(M)$. Then all metrics on M satisfy the bound*

$$\frac{\operatorname{stsys}_2^n}{\operatorname{vol}} \le n!\,(\Gamma_b)^n.$$

The conjecture holds in the following cases:

(1) in dimension 4 by Theorem 10.9;
(2) in the case $b_2(M)=1$ by Theorem 9.10.5.

CHAPTER 11

Further generalisations of Gromov's inequality

11.1. Obtimal stable systolic inequality for $b_2 = 2$

THEOREM 11.1.1. *In the hypotheses of Conjecture* 10.10.2, *if one has* $\dim M = 2n$ *and* $b_2(M) = 2$ *then every metric on* M *satisfies*

$$\frac{\mathrm{stsys}_2^n}{\mathrm{vol}} \leq n!\,(\Gamma_2)^n.$$

PROOF. Let $a, b \in \bigwedge_p^2 M$ be 2-forms of unit comass. By Wirtinger inequality, the comass norms of $a^{\wedge k}$ and $b^{\wedge(n-k)}$ are bounded by $k!$ and $(n-k)!$ respectively. An easy combinatorial argument shows that the rank of the form $a^{\wedge k}$ is bounded by $\binom{n}{k}$ and similarly for the form b. This provides a bound on the ratio of the Euclidean norm and the comass norm. The Hodge star estimate as in the proof of Theorem 10.9.4 produces the bound $|a^{\wedge k} \wedge b^{\wedge(n-k)}| \leq k!\,(n-k)!\,\binom{n}{k} = n!$ and we conclude as in the proof of Theorem 10.9.4 using the duality of the stable norm and the comass norm. $\square$

11.2. Inequalities for products of manifolds

We treat some consequences for specific classes of manifolds constructed from the ones already available from the first part of the course.

PROPOSITION 11.2.1. *Let* $n, m \geq 1$. *All metrics on the* $2(n+m)$-*manifold* $M = \mathbb{CP}^n \times \mathbb{CP}^m$ *satisfy*

$$\mathrm{stsys}_2(M)^{n+m} \leq (n+m)!\,(\Gamma_2)^{n+m}\,\mathrm{vol}(M).$$

PROOF. We have $b_2(M) = 2$ by Theorem 10.2.2. Let α denote the pullback (see Definition 5.6.2) class of the Fubini–Study 2-form of $\mathbb{CP}^n$. Let β be the pullback of the class of the Fubini–Study 2-form of $\mathbb{CP}^m$. Recall that the volume form of a product is the wedge product of the volume forms (see Example 9.6.3). Consider the class

$$\alpha^{\cup n} \cup \beta^{\cup m} \in H_{\mathrm{dR}}^{2(n+m)}(M). \tag{11.2.1}$$

The class (11.2.1) is proportional to the fundamental cohomology class of M, and in particular does not vanish. It follows that the required cup-product condition is satisfied, and the bound follows by Theorem 11.1.1. □

PROPOSITION 11.2.2. *Let $n \geq 1$. All metrics on the $2n$-dimensional manifold $M = S^2 \times \cdots \times S^2$ (n factors) satisfy*

$$\operatorname{stsys}_2(M) \leq C_n \sqrt[n]{\operatorname{vol}(M)}$$

for a suitable constant $C_n > 0$ independent of the metric.

PROOF. From the Künneth formula, we have $b_2(M) = n$ by Theorem 10.2.2. Let α_i be the class of the pullback to M (see Definition 5.6.2) of the area form of the i-th factor S^2. The product

$$\alpha_1 \cup \cdots \cup \alpha_n \in H^{2n}_{\mathrm{dR}}(M)$$

is nonzero, and, suitably normalized, represents the fundamental cohomology class of M. The inequality follows from Theorem 10.10.1. □

PROPOSITION 11.2.3. *Let Σ be an orientable surface. Then all metrics on $M = \Sigma \times S^2$ satisfy*

$$\operatorname{stsys}_2(M) \leq \sqrt{2}\, \Gamma_2 \sqrt{\operatorname{vol}(M)}.$$

PROOF. We have $b_2(M) = 2$ since in Theorem 10.2.2 simple connectivity is required of only one of the factors. One of the 2-dimensional classes is obtained by pullback from Σ, and the other by pullback from S^2. Their cup product is the fundamental cohomology class of M. Thus the cup-product condition is satisfied, and we apply Theorem 11.1.1. □

11.3. Optimal systolic inequality on $S^1 \times S^n$

All stable systolic inequalities considered so far involved only the stable 2-systole $\operatorname{stsys}_2(M)$. The techniques we developed apply more generally and yield inequalities involving other stable k-systoles. As an example, we consider the case of $M = S^1 \times S^n$ when $n \geq 2$.

THEOREM 11.3.1. *Let $n \geq 2$. All metrics on $M = S^1 \times S^n$ satisfy the inequality* $\operatorname{stsys}_1(M) \operatorname{stsys}_n(M) \leq \operatorname{vol}_{n+1}(M)$.

REMARK 11.3.2 (Unstable counterexamples). The inequality clearly holds for product metrics on M (see Section 9.11) even for the ordinary systoles (not the stable ones). One might have thought that the inequality $\operatorname{sys}_1(M) \operatorname{sys}_n(M) \leq \operatorname{vol}(M)$ (for the ordinary systoles) should hold for all metrics on M. It turns out that there are counterexamples, as discussed in [**12**] and Section 11.4.

Proof of Theorem 11.3.1. Let $\alpha \in H^1_{\mathrm{dR}}(M)$ be the pullback of the fundamental cohomology class of S^1. Let $\beta \in H^n_{\mathrm{dR}}(M)$ be the pullback of the fundamental cohomology class of S^n. Then $\alpha \cup \beta$ is the fundamental cohomology class of M. Let $a \in \alpha$ be a 1-form. Let $b \in \beta$ be an n-form. In dimension and codimension 1, all forms are simple. Therefore the comass and the Euclidean norm coincide, and we have as the point u ranges over M, by Cauchy–Schwarz,

$$\begin{aligned} 1 = \int_M \alpha \cup \beta &\le \int_M |a \wedge b|_u \, d\,\mathrm{vol}_M \\ &= \int_M \langle a, *b \rangle_u \, d\,\mathrm{vol}_M \\ &\le \int_M |a|_u \, |b|_u \, d\,\mathrm{vol}_M \\ &\le |a|_\infty \, |b|_\infty \, \mathrm{vol}(M). \end{aligned}$$

Minimizing over $a \in \alpha$ and $b \in \beta$, we obtain

$$1 \le \|\alpha\|^* \|\beta\|^* \,\mathrm{vol}(M) = \lambda_1 \left(L^1_{\mathrm{dR}}(M), \| \, \|^* \right) \, \lambda_1 \left(L^n_{\mathrm{dR}}(M), \| \, \|^* \right) \mathrm{vol}(M) \tag{11.3.1}$$

where $\| \, \|^*$ denotes the comass norms respectively in H^1_{dR} and in H^n_{dR}. Let $L^1 = L^1_{\mathrm{dR}}(M)$ and $L^n = L^n_{\mathrm{dR}}(M)$. Let $L_1 = H_1(M;\mathbb{Z})$ and $L_n = H_n(M;\mathbb{Z})$. Multiplying both sides of the inequality (11.3.1) by the product $\lambda_1(L_1, \| \, \|) \, \lambda_1(L_n, \| \, \|) = \mathrm{stsys}_1 \, \mathrm{stsys}_n$, we obtain

$$\begin{aligned} &\mathrm{stsys}_1(M) \, \mathrm{stsys}_n(M) \\ &\quad \le \Big[\lambda_1(L_1, \| \, \|) \, \lambda_1 \left(L^1, \| \, \|^* \right) \Big] \Big[\lambda_1(L_n, \| \, \|) \, \lambda_1 \left(L^n, \| \, \|^* \right) \Big] \, \mathrm{vol}(M). \end{aligned}$$

Now we use the duality of the stable norm and the comass norm. Since $b_1(M) = b_n(M) = 1$, the product of the λ_1's of dual lattices equals 1, proving the theorem. $\square$

11.4. Counterexample to systolic inequality on $S^1 \times S^n$

The behavior of ordinary (unstable) systoles is very different from that of stable systoles.[1]

THEOREM 11.4.1 (Gromov). *The manifold $S^1 \times S^3$ admits metrics with arbitrarily large ratio $\frac{\mathrm{sys}_1 \mathrm{sys}_3}{\mathrm{vol}}$.*

The proof appears below following Proposition 11.4.8. The unit sphere $S^3 \subseteq \mathbb{C}^2$ admits an action by complex scalars $e^{i\theta}$, namely $e^{i\theta}(z_1, z_2) = \big(e^{i\theta} z_1, e^{i\theta} z_2\big)$. In particular we have an action by the primitive nth root of unity,

$$\zeta_n = e^{\frac{2\pi i}{n}} \in \mathbb{C}. \tag{11.4.1}$$

LEMMA 11.4.2. *We have* $\mathrm{vol}(S^3) = 2\pi^2$.

PROOF. Apply Fubini's theorem to the Hopf fibration $S^3 \to \mathbb{CP}^1$, noting that the fiber has length 2π and $\mathbb{CP}^1$ has area π. □

DEFINITION 11.4.3. Consider the manifold $M = \mathbb{R} \times S^3$ with its product metric. We use the orthogonal projection to the second factor S^3 in $\mathbb{R} \times S^3$ to pull back the volume form of the 3-sphere to a 3-form $\alpha \in \Omega^3(M)$.

Consider the isometry of M which translates in the $\mathbb{R}$-direction by $\frac{1}{n^2}$ and spins the fiber of the Hopf fibration by $\frac{2\pi}{n}$:

DEFINITION 11.4.4. Consider the isometry

$$\begin{aligned} \tau_n \colon M &\to M \\ \tau_n(r, s) &= \big(r + \tfrac{1}{n^2},\, \zeta_n\, s\big), \end{aligned} \tag{11.4.2}$$

where ζ_n is as in (11.4.1).

The isometry τ_n generates an action of $\mathbb{Z} = \langle \tau_n \rangle$ on M.

DEFINITION 11.4.5. Let $Q_n = M / \langle \tau_n \rangle$ be the quotient manifold.

The following is immediate from the construction.

PROPOSITION 11.4.6. *The quotient manifold Q_n has the following properties:*

(1) *The natural product metric on $\mathbb{R} \times S^3$ descends to a metric g_n on Q_n.*[2]
(2) *The manifold Q_n is diffeomorphic to $S^1 \times S^3$ but the metric g_n is not a direct product of the component metrics.*

[1] Gromov [**Gro96**].
[2] which is no longer a direct product of metrics on the factors; see item (2).

(3) *The* 3*-form* α *is invariant under* τ_n *and therefore descends from* M *to* Q_n.
(4) *The form* $\alpha \in \Omega^3(Q_n)$ *cannot be obtained by pullback by a coordinate projection of a form on* $S^1 \times S^3$.
(5) *We have* $H_3(Q_n;\mathbb{Z}) \simeq \mathbb{Z}$.

PROOF. Item (1) follows from the fact that the translation (11.4.2) is an isometry. □

PROPOSITION 11.4.7. *The* 3*-systole of* Q_n *is* $2\pi^2$*, while the volume is* $\frac{2\pi^2}{n^2}$.

PROOF. Consider a 3-dimensional submanifold $C \subseteq Q_n$ representing a generator of $H_3(Q_n;\mathbb{Z}) \simeq \mathbb{Z}$. For any such C, by Stokes theorem we have $\int_C \alpha = 2\pi^2$ (the volume of the unit sphere S^3 as above). Since the 3-form α has unit comass, we obtain that $\mathrm{vol}_3(C) \geq 2\pi^2$ (such an argument is called a calibration argument). It follows that $\mathrm{sys}_3(Q_n) = 2\pi^2$. From (11.4.2) it follows that

$$\mathrm{vol}(Q_n) = \frac{2\pi^2}{n^2}, \tag{11.4.3}$$

as required. □

PROPOSITION 11.4.8. $\mathrm{sys}_1(Q_n)$ *is on the order of* $\frac{1}{n}$.

PROOF. We will use the characterisation of $\mathrm{sys}_1(Q_n)$ in terms of distances in M as follows:

$$\mathrm{sys}_1(Q_n) = \max\left\{ d\big((r,s), \tau_n^k(r,s)\big) \colon k \neq 0 \right\}.$$

For a point $(r,s) \in M$, the distance d between (r,s) and $\tau_n^k(r,s)$ is bounded below as follows:

$$d\big((r,s), \tau_n^k(r,s)\big) \geq \max\left\{ d\big(r, r + \tfrac{k}{n^2}\big),\, d(s, \zeta_n^k\, s)\right\}.$$

Thus $d\big(\tau_n(r,s),(r,s)\big) \geq \frac{2\pi}{n}$. Consider the orbit of the point (r,s) under the action of the isometry τ_n. Applying successive powers of the isometry τ_n, we obtain points in the manifold Q_n which initially get further and further away from the original point (r,s). The S^3-component of the point $\tau_n^k(r,s) \in M$ starts getting smaller as k gets past $\frac{n}{2}$, when the second coordinate of $\tau_n^k(r,s)$ reaches the antipodal point of the original point $s \in S^3$. The S^3-component equals 0 when $k = n$ by the definition of τ_n as in (11.4.2). But by then, the change in the $\mathbb{R}$-component of the point $\tau_n^k(r,s)$ has grown to $\frac{1}{n^2} \cdot n = \frac{1}{n}$, as required. □

PROOF OF THEOREM 11.4.1. Propositions 11.4.7 and 11.4.8 imply that the product $\mathrm{sys}_1 \mathrm{sys}_3$ is at least $\frac{1}{n}$ (up to a constant), whereas by (11.4.3) the volume tends to zero faster, namely as $\frac{1}{n^2}$. Therefore the ratio $\frac{\mathrm{sys}_1\,\mathrm{sys}_3}{\mathrm{vol}}$ for the manifold Q_n grows linearly in n, as required. □

Similar counterexamples can in fact be constructed for $S^1 \times S^n$ whenever $n \geq 2$. The limitation $n \geq 2$ leads to the question whether a systolic inequality exists for $S^1 \times S^1$. This is the subject of Chapter 12.

11.5. Case $b_2 = 3$

We will use the following notation.

DEFINITION 11.5.1. Let J_n be the all-ones matrix of size $n \times n$.

LEMMA 11.5.2. *All 2-forms a, b, c of comass at most 1 in $\mathbb{R}^6$ satisfy $|a \wedge b \wedge c| \leq 6$.*

PROOF. By linearity, it suffices to prove that for the standard symplectic form $\alpha = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4 + \omega_5 \wedge \omega_6$, one has $|\alpha \wedge b \wedge c| \leq 6$ for all 2-forms b, c of comass 1. Consider the endomorphism M_α of $\bigwedge^2(\mathbb{R}^6)$ sending b to $*(\alpha \wedge b)$. Then $\bigwedge^2(\mathbb{R}^6)$ decomposes into invariant subspaces $V + W$ where V is spanned by the 2-forms $\omega_1 \wedge \omega_2$, $\omega_3 \wedge \omega_4$, and $\omega_5 \wedge \omega_6$, whereas W is spanned by the remaining $\omega_i \wedge \omega_j$. The restriction of M_α to W is given by a permutation matrix, whereas the restriction to V is $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = J_3 - I_3$, with eigenvalues $-1, -1, 2$. Hence the spectral radius of M is 2. Finally, by the Cauchy-Schwarz inequality

$$|\alpha \wedge b \wedge c| = |\langle M_\alpha(b), z\rangle| \leq 2\,|b|\,|c| \leq 6\,\|b\|^*\,\|c\|^*,$$

as required.[3] □

COROLLARY 11.5.3. *All metrics on a 6-dimensional 2-essential manifold M satisfy*

$$\frac{\mathrm{stsys}_2^3}{\mathrm{vol}} \leq 3!\left(\Gamma_{b_2(M)}\right)^3.$$

PROOF. Arguing as before, it suffices to show that if a, b, c are 2-forms of comass 1 on $\mathbb{R}^6$ then $|a \wedge b \wedge c| \leq 6$. The result follows by Lemma 11.5.2. □

[3]In [**Gro81**, item 4.37, p. 60], one finds the following comment in the paragraph discussing Wirtinger's inequality: "$(2n)!/2^n$ ... est la meilleure constante pour la comasse d'un produit de n 2-formes quelconques." This was translated as follows in [**8**, item 4.37, p. 262]: "$(2n)!/2^n$ ... is the best constant for the comass of the product of n arbitrary 2-forms." Note that $6!/2^3$ is considerably larger than 6.

Proposition 11.5.4. *All metrics on an 8-dimensional 2-essential manifold M with $b_2(M) = 3$ satisfy*

$$\frac{\mathrm{stsys}_2^4}{\mathrm{vol}} \leq 4!\left(\Gamma_3\right)^4.$$

Proof. To apply the usual argument exploiting duality between stable norm and comass norm, we need to show that all 2-forms a, b, c of comass at most 1 in $\mathbb{R}^8$ satisfy $|a \wedge a \wedge b \wedge c| \leq 4!$. Choose an orthonormal basis to diagonalize a as $a = \sum_j \lambda_j \, \omega_{2j-1} \wedge \omega_{2j}$. Then $a \wedge a$ contains simple 4-forms only with coefficients $\lambda_i \lambda_j$ for $i \neq j$ (the coefficient λ_j^2 does not appear). As before, it follows by linearity that it suffices to prove the estimate in the case when a is the standard symplectic form α. We define an endomorphism $M_{\alpha^{\wedge 2}}$ of $\bigwedge^2(\mathbb{R}^8)$ by sending b to the 2-form $*(\alpha \wedge \alpha \wedge b)$. Then $\bigwedge^2(\mathbb{R}^8)$ decomposes into invariant subspaces $V + W$ where V is spanned by the 2-forms $\omega_1 \wedge \omega_2$, $\omega_3 \wedge \omega_4$, $\omega_5 \wedge \omega_6$, and $\omega_7 \wedge \omega_8$, whereas W is spanned by the remaining 2-forms $\omega_i \wedge \omega_j$. The action on V is represented by the matrix $2(J_4 - I_4)$, with eigenvalues twice $-1, -1, -1, 3$, while the action on W is a permutation matrix. It follows that the spectral radius of $M_{\alpha^{\wedge 2}}$ is 6. Finally,

$$|\alpha \wedge \alpha \wedge b \wedge c| \leq \left| \left\langle M_{\alpha^{\wedge 2}}(b), c \right\rangle \right| \leq 6\,|b|\,|c| \leq 24\,\|b\|^*\|c\|^*,$$

as required. □

CHAPTER 12

Loewner's inequality

In this chapter we prove Loewner's torus inequality. In Chapter 13 we will prove Pu's inequality for the real projective plane. A different proof of Pu's inequality via quaternions appears in Chapter 14.

12.1. Eisenstein integers, Hermite constant

The Eisenstein integers were already mentioned in Example 6.4.4.

DEFINITION 12.1.1. The lattice $L_E \subseteq \mathbb{R}^2 = \mathbb{C}$ of the *Eisenstein integers* (also known as the hexagonal lattice) is the lattice in $\mathbb{C}$ spanned by the elements 1 and the sixth root of unity.

REMARK 12.1.2. To visualize the Eisenstein lattice L_E, start with an equilateral triangle in $\mathbb{C}$ with vertices 0, 1, and $\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and construct a tiling[1] of the plane by repeatedly reflecting in all sides of the triangle. The Eisenstein integers are by definition the set of vertices of the resulting tiling.

DEFINITION 12.1.3. Let $b \in \mathbb{N}$. The *Hermite constant* γ_b is defined in one of the following two equivalent ways:

(1) The constant γ_b is the *square* of the biggest $\lambda_1(L)$ among all lattices $L \subseteq \mathbb{R}^n$ such that $\mathrm{vol}(\mathbb{R}^b/L) = 1$;
(2) γ_b is defined by the formula

$$\sqrt{\gamma_b} = \sup\left\{ \frac{\lambda_1(L)}{\mathrm{vol}(\mathbb{R}^b/L)^{1/b}} : L \subseteq (\mathbb{R}^b, |\ \ |) \right\}, \tag{12.1.1}$$

where the supremum is over all lattices $L \subseteq \mathbb{R}^b$ with a Euclidean norm $|\ \ |$.[2]

For $b = 2$, the lattice of Eisenstein integers realizes the supremum in (12.1.1). We will use the following result to prove Loewner's torus inequality with isosystolic defect.

[1]ritzuf

[2]Relation to packing: A lattice realizing the supremum in (12.1.1) may be thought of as the one realizing the densest packing (ariza hachi tzfufa) in $\mathbb{R}^b$ obtained by placing balls of radius $\frac{1}{2}\lambda_1(L)$ at the points of L.

PROPOSITION 12.1.4. *When $b = 2$, we have the following value for the Hermite constant: $\gamma_2 = \frac{2}{\sqrt{3}} = 1.1547\ldots$. The corresponding best lattice is similar to the lattice of the Eisenstein integers.*

This is immediate from Corollary 10.4.5.[3]

12.2. Loewner's inequality

Loewner's torus inequality relates the total area to the systole, i.e., least length of a noncontractible loop on the torus $(\mathbb{T}^2, \mathbf{g})$. Loewner's inequality first appeared in [**Pu52**].

THEOREM 12.2.1 (Loewner's torus inequality). *Every metric $\mathbf{g}$ on the torus satisfies*

$$(\mathrm{sys}_1(\mathbf{g}))^2 \leq \gamma_2 \,\mathrm{area}(\mathbf{g}). \tag{12.2.1}$$

We will use an equivalent formulation $\mathrm{area}(\mathbf{g}) - \gamma_2^{-1} \,\mathrm{sys}_1(\mathbf{g})^2 \geq 0$ which is more easily generalized.

THEOREM 12.2.2 (Boundary case of equality). *The boundary case of equality in* (12.2.1) *is attained if and only if the metric is similar to the flat metric obtained as the quotient of $\mathbb{R}^2$ by the lattice formed by the Eisenstein integers.*

Recall the following.

(1) The 1-systole of a Riemannian manifold M is the least length of a noncontractible loop on M; see Section 9.2.
(2) The fundamental group of the torus $\mathbb{T}^2$ is abelian.
(3) Due to the multiplicavity of the volume of 1-homology classes for orientable surfaces (see Section 8.14), we have

$$\mathrm{sys}_1(\mathbb{T}^2, \mathbf{g}) = \lambda_1 \left(H_1(\mathbb{T}^2; \mathbb{Z}), \| \, \| \right),$$

where $\| \, \|$ is the stable norm of the metric $\mathbf{g}$.

In this sense, Gromov's inequality (9.14.7) for the complex projective plane is a higher-dimensional analogue of Loewner's inequality.

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[3]In more detail, let $L \subseteq \mathbb{C}$. Multiplying the lattice L by nonzero complex numbers does not change the value of the scale-invariant quotient $\frac{\lambda_1(L)^2}{\mathrm{area}(\mathbb{C}/L)}$ occurring in the definition of the Hermite constant. Thus we may assume that $\lambda_1(L) = 1$, and moreover $1 \in L$. Thus instead of proving the upper bound for λ_1, it suffices to normalize it to 1 and prove a lower bound for the area. We choose τ as in Lemma 10.4.3. Since τ is in the fundamental domain, the imaginary part of τ satisfies $\mathrm{Im}(\tau) \geq \frac{\sqrt{3}}{2}$, with equality only when $\tau = e^{i\frac{2\pi}{3}} \in D_0$. The area of the parallelogram in $\mathbb{C}$ spanned by and $+1$ and τ is its altitude. The altitude is the imaginary part $\mathrm{Im}(\tau) \geq \frac{\sqrt{3}}{2}$. It follows that the ratio $\frac{\lambda_1(L)^2}{\mathrm{area}(\mathbb{C}/L)}$ is at most $\frac{2}{\sqrt{3}}$. Alternatively, one can use Corollary 10.4.5.

12.3. Bonnesen's inequality

We outline an analogy between Loewner's inequality and the isoperimetric inequality.[4] Bonnesen's inequality asserts the following strengthening of the isoperimetric inequality.

THEOREM 12.3.1 (Bonnesen's inequality). *Consider a Jordan curve of length L in the plane. Let A be the area of the region bounded by the curve. Let R be the circumradius of the bounded region, and let r be its inradius. Then*

$$L^2 - 4\pi A \geq \pi^2(R-r)^2. \tag{12.3.1}$$

The inequality first appeared in [**Bo21**].[5] The remainder or "error" term $\pi^2(R-r)^2$ on the right hand side of (12.3.1) is traditionally called the *isoperimetric defect term*.[6]

Loewner's torus inequality (12.2.1) can be strengthened by introducing a "systolic defect" term à la Bonnesen. To express such an improvement of Loewner's inequality, we need to review the conformal representation theorem (uniformisation theorem) already discussed in in Section 3.4.4.

THEOREM 12.3.2 (Conformal representation theorem). *Every metric $\mathbf{g}$ on the torus $\mathbb{T}^2$ is isometric to a metric of the form*

$$f^2(x,y)(dx^2+dy^2), \tag{12.3.2}$$

with respect to a unit area flat metric $\mathbf{g}_0 = dx^2 + dy^2$ on the torus $\mathbb{C}/L$, for a suitable lattice $L \subseteq \mathbb{C}$.

See (3.4.10) for more details. The defect term in the strengthened Loewner inequality is the variance[7] of the conformal factor f in (12.3.2), as in Theorem 12.4.4 below.

12.4. Expected value and variance

Consider a flat torus $(\mathbb{T}^2, \mathbf{g}_0) = \mathbb{C}/L$ where L is a lattice and $\mathbf{g}_0 = dx^2 + dy^2$, normalized in such a way that $\text{area}(\mathbb{C}/L, \mathbf{g}_0) = 1$. The torus can be thought of as the quotient of the plane by for the action of L on $\mathbb{C}$ by translations.

DEFINITION 12.4.1. An open domain $D \subseteq \mathbb{C}$ is called an *open fundamental domain* for the torus if D is a polygon[8] mapping injectively to $\mathbb{T}^2$ while the closure $\bar{D}$ maps surjectively to $\mathbb{T}^2$.

---

[4] See further in in footnote 14 in Section 14.

[5] See [**BuraZ80**, p. 3] for a proof.

[6] she'erit izoperimetrit

[7] shonut

[8] metzula

Such fundamental domains will be used in the proof of Loewner's torus inequality with a remainder term in Section 12.7.

Definition 12.4.2. Let $D \subseteq \mathbb{C}$ be an open fundamental domain for the torus, and f a function on the torus. The mean, or expected value,[9] of f is the quantity

$$m = E(f) = \int_{\mathbb{T}^2} f\, dx \wedge dy = \int_D f(x,y)\, dx \wedge dy. \tag{12.4.1}$$

where $dx \wedge dy$ is the standard area form of $\mathbb{C}$.

The proof of inequalities with isosystolic defect relies upon the computational formula for the variance[10] of a random variable[11] in terms of expected values.

Definition 12.4.3. The *variance* of f is $\mathrm{Var}(f) = E_\mu\left((f-m)^2\right)$, where $m = E_\mu(f)$ is the expected value, i.e., the mean.

We have the following strengthening of Loewner's torus inequality.

Theorem 12.4.4 ([**Horowitz et al.**]). *Every metric* $\mathbf{g}$ *on the torus satisfies the inequality*

$$\mathrm{area}(\mathbf{g}) - \gamma_2^{-1}\, \mathrm{sys}(\mathbf{g})^2 \geq \mathrm{Var}(f). \tag{12.4.2}$$

Here the remainder term on the right-hand side is the variance of the conformal factor f *of the metric* $\mathbf{g} = f^2(dx^2+dy^2)$ *on the torus, relative to the mesure induced by the unit area flat metric* $\mathbf{g}_0 = dx^2+dy^2$ *in the conformal class of* $\mathbf{g}$.

The nonzero remainder term on the right-hand side of (12.4.2) is analogous to the isoperimetric defect of Bonnesen's inequality.

12.5. Application of computational formula for variance

Definition 12.5.1. Let μ be a *probability measure* on a space M, meaning that the total measure of M is 1. The *computational formula for the variance* of a random variable f on M is the formula

$$E_\mu(f^2) - \left(E_\mu(f)\right)^2 = \mathrm{Var}(f). \tag{12.5.1}$$

In our differential geometric application, the random variable f is the conformal factor of the metric on the torus.

[9]tochelet

[10]shonut

[11]mishtaneh mikri? randomali?

DEFINITION 12.5.2. Let $\mathbf{g}_0 = dx^2 + dy^2$ be a flat metric of unit area on the 2-torus $\mathbb{T}^2 = \mathbb{C}/L$ (then the lattice L is said to be of unit coarea).

Denote the associated measure on $\mathbb{T}^2$ by μ, so that $\mu(\mathbb{T}^2) = 1$. The measure coincides with the usual area for the kind of domains we are interested in. In other words, for every domain $A \subseteq \mathbb{T}^2$, we have $\mu(A) = \int_A dx \wedge dy \geq 0$. Since μ is a probability measure, we can apply the computational formula for the variance, (12.5.1), to μ.

REMARK 12.5.3. Here f can be thought of either as a function on the torus or as a doubly periodic function on $\mathbb{C}$ (i.e., periodic with respect to translations by elements of the lattice L).

LEMMA 12.5.4. *Consider a metric $\mathbf{g} = f^2\mathbf{g}_0$ on the torus conformal to the flat metric $\mathbf{g}_0$ of unit area, where $f > 0$ is the conformal factor. Then we have*

$$E_\mu(f^2) = \int_{\mathbb{T}^2} f^2 dx \wedge dy = \mathrm{area}(\mathbf{g}).$$

PROOF. Indeed, $f^2 dx \wedge dy$ is the area 2-form of the metric $\mathbf{g}$. $\square$

In our case, the computational formula for the area (12.5.1) therefore becomes

$$\mathrm{area}(\mathbf{g}) - \left(E_\mu(f)\right)^2 = \mathrm{Var}(f). \tag{12.5.2}$$

In Section 12.6, we will relate the expected value $E_\mu(f)$ to the systole of the metric $\mathbf{g}$. Then we will relate formula (12.5.2) to Loewner's torus inequality in Section 12.7.

12.6. Basis for lattice of unit coarea

Consider the torus $\mathbb{T}^2 = \mathbb{C}/\tilde{L}$ with the metric $\mathbf{g}_0 = dx^2 + dy^2$ of unit area. By Lemma 12.1.4, the lattice $\tilde{L}$ of deck transformations of the flat torus $(\mathbb{T}^2, \mathbf{g}_0)$ admits a $\mathbb{Z}$-basis similar to the basis $(1, \tau) \subseteq \mathbb{C}$, where $\tau \in D_0$ is the conformal parameter as in formula (10.4.1). In other words, $\tilde{L}$ is similar to the lattice $L = \mathbb{Z}\,1 + \mathbb{Z}\,\tau \subseteq \mathbb{C}$, where $\tau(L)$ is the conformal parameter of L or any similar lattice.

DEFINITION 12.6.1. We set $\sigma = \sqrt{\mathrm{Im}\big(\tau(L)\big)} > 0$, where $\mathrm{Im}(\tau)$ is the imaginary part of the conformal parameter $\tau \in D_0$.

LEMMA 12.6.2. *We have $\sigma^2 \geq \gamma_2^{-1}$, with equality if and only if the conformal parameter τ is the primitive cube root of unity $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$.*

PROOF. This is immediate from the geometry of the fundamental domain. $\square$

Lemma 12.6.3. *The basis for the lattice $\tilde{L}$ (group of deck tranformations of $\mathbf{g}_0$) can be taken to be $(\sigma^{-1}, \sigma^{-1}\tau)$, where $\operatorname{Im}(\sigma^{-1}\tau) = \sigma$.*

Proof. This is immediate from the fact that $\mathbf{g}_0$ is of unit area. □

12.7. Proof of Loewner's torus inequality

We start with the following data.

(1) a lattice $L \subseteq \mathbb{C}$ with conformal parameter $\tau = \tau(L)$, where $L = \operatorname{Span}_{\mathbb{Z}}(1, \tau)$;
(2) $\sigma = \sqrt{\operatorname{Im}(\tau(L))}$;
(3) $\tilde{L} = \frac{1}{\sigma}L$ is a lattice of unit coarea similar to L, with basis $\left(\frac{1}{\sigma}, \frac{\tau}{\sigma}\right)$;
(4) $\mathbb{T}^2 = \mathbb{C}/\tilde{L}$ is a flat torus of unit area, denoted $\mathbf{g}_0$.

We add the following two items.

Definition 12.7.1. We define a family of horizontal geometrics and a fundamental domain for the metric $\mathbf{g}_0$:

(1) $\mathbf{g}_0$ is ruled by a pencil of horizontal geodesics η_y where $y \in [0, \sigma]$ (see Section 12.6);
(2) we have an open fundamental domain B for the torus, given by the rectangle $B = (0, \sigma^{-1}) \times (0, \sigma)$ in the (x, y)-plane.

Lemma 12.7.2. *For the metric $\mathbf{g} = f^2\mathbf{g}_0$ on the torus, we have the following lower bound for the lengths of closed geodesics η_y:*

$$\forall y \in [0, \sigma], \ \operatorname{length}_{\mathbf{g}}(\eta_y) \geq \operatorname{sys}_1(\mathbf{g}). \tag{12.7.1}$$

Proof. This is immediate from the fact that each of these closed geodesics is noncontractible[12] in $\mathbb{T}^2$. □

From (12.5.2), we have

$$\operatorname{area}(\mathbf{g}) - \left(E_\mu(f)\right)^2 = \operatorname{Var}(f); \tag{12.7.2}$$

We now analyze the expected value term $E_\mu(f) = \int_{\mathbb{T}^2} f dx \wedge dy$. We will use Lemma 12.7.2 in the proof of the following proposition. Recall that $\sigma = \sqrt{\operatorname{Im}\tau(L)}$.

Proposition 12.7.3. *The metric $\mathbf{g} = f^2(dx^2 + dy^2)$ on the torus satisfies $E_\mu(f) \geq \sigma \operatorname{sys}_1(\mathbf{g})$.*

[12] lo-kvitza, with kaf

PROOF. Let B be the fundamental domain for the torus as Definition 12.7.1. By Fubini's theorem, we pass to the iterated integral[13] to obtain the following lower bound for the expected value:

$$\begin{aligned} E_\mu(f) &= \int_B f(x,y)dx \wedge dy \\ &= \int_0^\sigma \left(\int_{\eta_y} f(x,y)dx \right) dy \\ &= \int_0^\sigma \mathrm{length}_{\mathbf{g}}(\eta_y) dy \\ &\geq \sigma \,\mathrm{sys}_1(\mathbf{g}), \end{aligned}$$

by inequality (12.7.1) of Lemma 12.7.2. □

COROLLARY 12.7.4. *Every metric* $\mathbf{g}$ *on* $\mathbb{T}^2$ *with conformal parameter* $\tau \in D_0$ *satisfies*

$$\mathrm{area}(\mathbf{g}) - \mathrm{Im}(\tau)\,\mathrm{sys}_1(\mathbf{g})^2 \geq \mathrm{Var}(f). \tag{12.7.3}$$

where f *is the conformal factor with respect to the unit area flat metric* $\mathbf{g}_0$, *i.e.,* $\mathbf{g} = f^2 \mathbf{g}_0$.

PROOF. Recall that $\sigma = \sqrt{\mathrm{Im}(\tau)}$. By Proposition 12.7.3 we have an inequality

$$E_\mu(f) \geq \sigma \,\mathrm{sys}_1(\mathbf{g}).$$

Substituting this inequality into the formula (12.5.2), we obtain the inequality $\mathrm{area}(\mathbf{g}) - \sigma^2 \mathrm{sys}_1(\mathbf{g})^2 \geq \mathrm{Var}(f)$, as required. □

Now by Lemma 12.6.2, we have $\sigma^2 \geq \gamma_2^{-1}$. Therefore we obtain in particular Loewner's torus inequality with isosystolic defect, i.e., Theorem 12.4.4:

COROLLARY 12.7.5. *Every metric* $\mathbf{g}$ *on the torus satisfies*

$$\mathrm{area}(\mathbf{g}) - \gamma_2^{-1}\,\mathrm{sys}_1(\mathbf{g})^2 \geq \mathrm{Var}(f). \tag{12.7.4}$$

12.8. Capacity of annuli

We connect our analysis of the torus $\mathbb{C}/L$ to the notion of the capacity of the associated annulus[14] (cylinder).

DEFINITION 12.8.1. The annulus A (i.e., cylinder) can be described in three equivalent ways:

[13]Integral nishneh

[14]tabaat (tet, bet, ayin, tav)

(1) A is obtained by cutting the torus $\mathbb{C}/L$ along the loop corresponding to the generator 1 of the pair $(1, \tau)$.
(2) A is obtained from the parallelogram spanned by the elements 1 and τ by identifying the sides parallel to τ.
(3) Let $\mathbb{R}/\mathbb{Z}$ be the circle of length 1. Then A is isometric to $\mathbb{R}/\mathbb{Z} \times [0, \sigma^2)$.

DEFINITION 12.8.2. The *capacity* $cap(A)$ of the cylinder is the inverse of its "height" $\mathrm{Im}(\tau) = \sigma^2$:

$$cap(A) = \frac{1}{\sigma^2}.$$

Note the the capacity is a conformal invariant. In this terminology, inequality (12.7.3) implies the following.

COROLLARY 12.8.3. *Every metric on the annulus A satisfies*

$$\frac{\mathrm{sys}_1^2}{\mathrm{area}} \le cap(A).$$

Thus we obtain the following corollary for an arbitrary surface (not necessarily a torus).

COROLLARY 12.8.4. *Let M be a surface. Assume the following:*

(1) *M contains an annulus $A \subseteq M$ conformal to $\mathbb{R}/\mathbb{Z} \times [0, \sigma^2)$.*
(2) *the circle $\mathbb{R}/\mathbb{Z} \times \{0\} \subseteq M$ is noncontractible in M.*

Then every metric on M satisfies the equivalent inequalities

$$\begin{cases} \mathrm{area}(M) \ge \sigma^2 \, \mathrm{sys}_1^2(M) \\ \mathrm{sys}_1^2(M) \le \mathrm{cap}(A) \, \mathrm{area}(M). \end{cases}$$

PROOF. The argument is similar to the proof of Loewner's inequality. The area of the surface is greater than or equal to the area of the annulus:

$$\mathrm{area}(M) \ge \mathrm{area}(A).$$

We apply the Cauchy–Schwarz inequality to the conformal factor f where the metric on the annulus is $f^2(dx^2 + dy^2)$ where $0 < x < 1$ and $0 < y < \sigma$, and then integrate by Fubini's theorem. This results in an inequality

$$\mathrm{area}(A) \ge \sigma^2 \, \mathrm{sys}_1^2(A)$$

as before. Hence $\mathrm{area}(M) \ge \mathrm{area}(A) \ge \sigma^2 \, \mathrm{sys}_1^2(A) \ge \sigma^2 \, \mathrm{sys}_1^2(M)$. $\square$

12.9. Boundary case of equality in Loewner's inequality

COROLLARY 12.9.1. *A metric* $\mathbf{g}$ *on the torus satisfying the boundary case of equality in Loewner's torus inequality*

$$\operatorname{area}(\mathbf{g}) - \gamma_2^{-1} \operatorname{sys}_1(\mathbf{g})^2 \geq 0 \tag{12.9.1}$$

is necessarily flat and similar to the quotient of $\mathbb{C}$ *by the lattice of Eisenstein integers.*

PROOF. If a metric $\mathbf{g} = f^2(dx^2 + dy^2)$ satisfies the boundary case of equality in (12.9.1), then the variance of the conformal factor f must vanish by (12.7.4). Hence f is a constant function. The proof is completed by applying Lemma 12.1.4 on the Hermite constant in dimension 2, taking into account the fact that the lattice L_E of Eisenstein integers represents the only conformal class satisfying the equality $\operatorname{Im}(\tau) = \gamma_2^{-1}$. □

12.9.1. Rectangular lattices and tori of revolution. This material is optional. Suppose $\tau(L)$ is pure imaginary, *i.e.* the lattice is a rectangular lattice. Let $\mathbf{g}_0$ be the corresponding flat torus $\mathbb{C}/L$.

COROLLARY 12.9.2. *If* τ *is pure imaginary, then the metric* $\mathbf{g} = f^2\mathbf{g}_0$ *satisfies the inequality* $\operatorname{area}(\mathbf{g}) - \operatorname{sys}_1(\mathbf{g})^2 \geq \operatorname{Var}(f)$.

PROOF. If τ is pure imaginary then $\sigma = \sqrt{\operatorname{Im}(\tau)} \geq 1$, and the inequality follows from (12.7.3). □

COROLLARY 12.9.3. *Every torus of revolution satisfies the inequality* $\operatorname{area}(\mathbf{g}) - \operatorname{sys}_1(\mathbf{g})^2 \geq \operatorname{Var}(f)$.

PROOF. This is immediate from the fact that its lattice is rectangular by Corollary 3.4.29. □

Note that this inequality for tori of revolution could not be optimal because the conformal factor f cannot be constant. Indeed, there is no embedding of a flat torus in $\mathbb{R}^3$.

12.9.2. Miscellaneous remarks. This material is optional. Perhaps the most familiar physical manifestation of the 3-dimensional isoperimetric inequality is the shape of a drop of water. Namely, a drop will typically assume a symmetric round shape. Since the amount of water in a drop is fixed, surface tension forces the drop into a shape which minimizes the surface area of the drop, namely a round sphere. Thus the round shape of the drop is a consequence of the phenomenon of surface tension (metach panim). Mathematically, this phenomenon is expressed by the isoperimetric inequality in the plane. The solution to the isoperimetric problem in the plane is usually expressed in the form of an inequality that relates the length L of a closed curve and the area A of the planar region that it encloses. The isoperimetric inequality states that $4\pi A \le L^2$, and that the equality holds if and only if the curve is a round circle. The inequality is an upper bound for area in terms of length. It can be rewritten as follows: $L^2 - 4\pi A \ge 0$. Recall the notion of central symmetry: a Euclidean polyhedron is called centrally symmetric if it is invariant under the "antipodal" map $x \mapsto -x$. Thus, in the plane central symmetry is the rotation by 180 degrees. For example, an ellipse is centrally symmetric, as is any ellipsoid in 3-space. There is a geometric inequality that is in a sense dual to the isoperimetric inequality in the following sense. Both involve a length and an area. The isoperimetric inequality is an upper bound for area in terms of length. There is a geometric inequality which provides an upper bound for a certain length in terms of area. More precisely it can be described as follows. Any centrally symmetric convex body of surface area A can be squeezed through a noose of length $\sqrt{\pi A}$, with the tightest fit achieved by a sphere. This property is equivalent to a special case of Pu's inequality (see below), one of the earliest systolic inequalities. An ellipsoid is an example of a convex centrally symmetric body in 3-space. It may be helpful to the reader to develop an intuition for the property mentioned above in the context of thinking about ellipsoidal examples. An alternative formulation is as follows. Every convex centrally symmetric body P in $\mathbb{R}^3$ admits a pair of opposite (antipodal) points and a path of length L joining them and lying on the boundary ∂P of P, satisfying $L^2 \le \frac{\pi}{4}\mathrm{area}(\partial P)$. This material is optional and is a review of Section 9.2. The *systole* of a compact metric space X is a metric invariant of X, defined to be the least length of a noncontractible loop in X, denoted $\mathrm{sys}(X)$. When X is a graph, the invariant is usually referred to as the *girth*, ever since the 1947 article by W. Tutte [**Tu47**]. Possibly inspired by Tutte's article, Loewner started thinking about systolic questions on surfaces in the late 1940s, resulting in a 1950 thesis by his student P.M. Pu [**Pu52**]. The actual term "systole" itself was not coined until a quarter century later, by Marcel Berger. This line of research was, apparently, given further impetus by a remark of René Thom, in a conversation with Berger in the library of Strasbourg University during the 1961-62 academic year, shortly after the publication of the papers of R. Accola and C. Blatter. Referring to these systolic inequalities, Thom reportedly exclaimed: "Mais c'est fondamental!" [These results are of fundamental importance!] Subsequently, Berger popularized the subject in a series of articles and books.[15] Systolic geometry features a number of recent publications in leading journals. Recently, an intriguing link has emerged with the Lusternik-Schnirelmann category. The existence of such a link can be thought of as a theorem in "systolic topology"; see [**Ka07**].

[15]See e.g., Berger [**Ber08**].

CHAPTER 13

Pu's inequality and generalisations

We will prove a generalisation of Pu's inequality for the real projective plane. A different proof of Pu's inequality via quaternions appears in Chapter 14.[1]

13.1. Statement of Pu's inequality

Pu's inequality applies to arbitrary Riemannian metrics on the real projective plane $\mathbb{RP}^2$. A student of Charles Loewner's, P.M. Pu proved it in a 1950 thesis (published in 1952 as [**Pu52**]). Pu's inequality is analogous to Loewner's torus inequality of Section 12.2.

THEOREM 13.1.1 (Pu). *every metric on the real projective plane* $\mathbb{RP}^2$ *satisfies*

$$\frac{\mathrm{sys}_1^2}{\mathrm{area}} \le \frac{\pi}{2}.$$

The case of equality is attained precisely by the metrics of constant Gaussian curvature on $\mathbb{RP}^2$.

There is a vast generalisation of the inequalities of Pu and Loewner, due to M. Gromov, called Gromov's systolic inequality for essential manifolds (different from Gromov's inequality for $\mathbb{CP}^n$ as in Theorem 9.10.4). This result involves a topological notion of essential manifold as in Section 15.1.[2]

[1]See [**6**].

[2]Loewner's systolic inequality for the torus and Pu's inequality for the real projective plane were historically the first results in systolic geometry. Great stimulus was provided in 1983 by Gromov's paper [**4**], and later by his book [**8**]. Our goal is to prove a strengthened version with a remainder term of Pu's systolic inequality $\mathrm{sys}^2(g) \le \frac{\pi}{2}\,\mathrm{area}(g)$ (for an arbitrary metric g on $\mathbb{RP}^2$), analogous to Bonnesen's inequality $L^2 - 4\pi A \ge \pi^2(R-r)^2$, where L is the length of a Jordan curve in the plane, A is the area of the region bounded by the curve, R is the circumradius and r is the inradius. Note that both the original proof in Pu ([**Pu52**], 1952) and the one given by Berger ([**1**], 1965, pp. 299–305) proceed by averaging the metric and showing that the averaging process decreases the area and increases the systole. Such an approach involves a 5-dimensional integration (instead of a 3-dimensional

13.2. Tangent map

To introduce the notion of a Riemannian submersion[3] in Section 13.3, we will need the notion of the tangent map of a smooth map ϕ between manifolds. Recall that we have the following proposition (see Proposition 5.6.1).

PROPOSITION 13.2.1. *A smooth map $\phi\colon M \to N$ between differentiable manifolds defines a natural map*

$$d\phi\colon TM \to TN$$

called the tangent map.

REMARK 13.2.2 (Relation to differential). If $N = \mathbb{R}$, then the tangent space to N at each point is naturally identified with $\mathbb{R}$ itself, and we obtain the notion of differential df of a smooth function $f\colon M \to \mathbb{R}$ as in Section 4.2.

13.3. Riemannian submersions

In this section we will define the notion of a Riemannian submersion. Consider a map $\phi\colon M \to N$ between closed manifolds. We assume that $d\phi$ is onto. Then by the implicit function theorem, the fibers are smooth submanifolds. Let $F \subseteq M$ be the fiber over a point $p \in N$.

PROPOSITION 13.3.1. *The kernel of the tangent map $d\phi\colon TM \to TN$ is the subspace $T_xF \subseteq TM$ at every point $x \in F$.*

PROOF. A vector tangent to the fiber can be represented by a path lying entirely in the fiber. □

DEFINITION 13.3.2. The *vertical space*[4] in T_xM is the subspace

$$\ker(d\phi_x) = T_xF.$$

Now we assume that M is equipped with a Riemannian metric.

DEFINITION 13.3.3. Given a Riemannian metric on M, the *horizontal space $H_x \subseteq T_xM$* is the orthogonal complement of the vertical space T_xF in T_xM.

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one given here) and makes it harder to obtain an explicit expression for a remainder term. Analogous results for the torus were obtained in [**Horowitz et al.**] with generalisations in [**BCIK04**], [**5**], [**7**]; see Chapter 12.

[3]Hatzafah according to Amit Solomon at `http://ma.huji.ac.il/~amit/hebrew_geometric_dictionary_2.pdf`

[4]merchav anchi

Here "H" stands for *horizontal*. Thus we have an orthogonal decomposition

$$T_x M = T_x F + H_x.$$

We now consider the restricted map

$$d\phi\restriction_{H_x} : H_x \to T_{\phi(x)} N$$

which we will also denote $d\phi$ for short.

DEFINITION 13.3.4. A map $\phi\colon M \to N$ between Riemannian manifolds is a *Riemannian submersion* if at every point $x \in M$, the restricted map $d\phi\colon H_x \to T_{\phi(x)} N$ is an isometry, i.e., preserves the length of vectors.

13.4. A Stiefel manifold

To prove Pu's inequality, we will exploit a special closed 3-dimensional manifold $M \subseteq \mathbb{R}^6$ called Stiefel manifold.

DEFINITION 13.4.1. The *Stiefel manifold* M is

$$M = \left\{ (v, w) \in \mathbb{R}^3 \times \mathbb{R}^3 \colon\ v \cdot v = 1,\ w \cdot w = 1,\ v \cdot w = 0 \right\} \qquad (13.4.1)$$

where $v \cdot w$ is the scalar product on $\mathbb{R}^3$.

LEMMA 13.4.2. *The Stiefel manifold M is diffeomorphic to the Lie group* $\mathrm{SO}(3, \mathbb{R})$ *of orthogonal three by three matrices of determinant* 1*.*

PROOF. We think of $\mathrm{SO}(3, \mathbb{R})$ is the space of matrices of unit determinant, with orthonormal column vectors. Let $n = v \times w$ be the vector product on $\mathbb{R}^3$. We define a diffeomorphism ϕ by setting

$$\phi\colon M \to \mathrm{SO}(3, \mathbb{R}), \quad (v, w) \mapsto (v \ \ w \ \ n).$$

The map ϕ is injective because it has a left inverse given by sending an orthogonal matrix $(a\ b\ c)$ of determinant 1 to the pair of vectors (a, b).

Given a matrix $P \in \mathrm{SO}(3, \mathbb{R})$ with first two columns v and w, the only possibilities for P are

$$(v \ \ w \ \ v{\times}w) \quad \text{and} \quad (v \ \ w \ \ -v{\times}w).$$

Only the first possibility has positive determinant. Hence

$$P = (v \ \ w \ \ v{\times}w) = \phi(v, w).$$

Therefore ϕ is onto. $\square$

PROPOSITION 13.4.3. *Given a point $(v, w) \in M$, the tangent space $T_{(v,w)} M$ is described by three conditions as follows:*

$$T_{(v,w)} M = \left\{ (X, Y) \in \mathbb{R}^3 \times \mathbb{R}^3 \colon X \cdot v = 0,\ Y \cdot w = 0,\ X \cdot w + Y \cdot v = 0 \right\}.$$

PROOF. The tangent space is identified by differentiating the three defining equations of M from (13.4.1) along a path through (v, w) with initial tangent vector (X, Y). $\square$

13.5. A metric on the Stiefel manifold

We define a Riemannian metric on M as follows. Given a point $(v, w) \in M$, let $n = v \times w$. We declare the basis

$$\big((0, n),\ (n, 0),\ (w, -v)\big)$$

of $T_{(v,w)}M$ to be orthonormal. This metric is a modification of the metric restricted to M from $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$. Namely, with respect to the Euclidean metric on $\mathbb{R}^6$ the above three vectors are orthogonal and the first two have length 1. However, the third vector has Euclidean length $\sqrt{2}$, whereas we need to define its length to be 1. Thus the metric on M is defined as follows.

Let $(v, w) \in M$, and let $n = v \times w$. Let $A \subseteq T_{(v,w)}M$ be the span of $(0, n)$ and $(n, 0)$, and let $B \subseteq T_{(v,w)}M$ be spanned by $(w, -v)$. We view the Euclidean metric $\mathbf{g}$ on $\mathbb{R}^6$ as a quadratic form.

DEFINITION 13.5.1. The metric $\mathbf{g}_M$ on M is obtained from the Euclidean metric $\mathbf{g}$ by setting

$$\mathbf{g}_M = \mathbf{g}\restriction_A \,+\, \tfrac{1}{2}\,\mathbf{g}\restriction_B \,. \tag{13.5.1}$$

THEOREM 13.5.2. *The vectors $(n, 0)$, $(0, n)$ and $(w, -v)$ form an orthonormal basis for $T_{(v,w)}M$ relative to the metric $\mathbf{g}_M$.*

PROOF. Our choice of the metric $\mathbf{g}_M$ ensures that $(w, -v)$ is a unit vector. $\square$

DEFINITION 13.5.3 (A pair of projections). The natural projections $p, q \colon M \to S^2$ are given by $p(v, w) = v$ and $q(v, w) = w$.

Each of the projections exhibits M as a circle bundle over S^2.

LEMMA 13.5.4. *The maps p and q on $(M, \mathbf{g}_M)$ are Riemannian submersions, where the metric on S^2 is restricted from $\mathbb{R}^3$.*

PROOF. For the projection p, given $(v, w) \in M$, the vector $(0, n)$ as defined above is tangent to the fiber $p^{-1}(v)$. Hence $dp(0, n) = 0$.

The other two vectors, $(n, 0)$ and $(w, -v)$, are thus an orthonormal basis for the horizontal subspace (see Section 13.3) of $T_{(v,w)}M$ normal to the fiber, and are mapped by dp to the orthonormal basis n, w of T_vS^2. $\square$

REMARK 13.5.5. The projection p maps the fiber $q^{-1}(w)$ onto a great circle $C \subseteq S^2$.

This map preserves length since the unit vector $(n, 0)$, tangent to the fiber $q^{-1}(w)$ at (v, w), is mapped by dp to the unit vector $n \in T_v S^2$. The same comments apply when the roles of p and q are reversed.

13.6. A double integration

In the proposition below, integration takes place respectively over great circles $C \subseteq S^2$, over the fibers in M, over S^2, and over M. The integration is always with respect to the volume element of the given Riemannian metric. We will use the following generalisation of Fubini's theorem.

THEOREM 13.6.1 (Fubini's theorem). *Given a Riemannian submersion $M \to N$, the integral over M can be computed by performing two successive integrations:*

(1) *integrating over each fiber;*
(2) *integrating the result of* (1) *over the base N.*

Since p and q are Riemannian submersions by Lemma 13.5.4, we can use Fubini's Theorem to integrate over M by integrating first over the fibers of either p or q, and then over S^2; cf. [**2**, Lemma 4].

REMARK 13.6.2. By the remarks above, if $C = p(q^{-1}(w))$ and a function $f \colon S^2 \to \mathbb{R}$ is continuous, then $\int_{q^{-1}(w)} f \circ p = \int_C f$.

PROPOSITION 13.6.3. *Given a function $f \colon S^2 \to \mathbb{R}^+$, we define the least mean $m \in \mathbb{R}$ by setting*

$$m = \min \left\{ \int_C f \colon \quad C \subseteq S^2 \ \textit{a great circle} \right\}.$$

Then

$$\frac{m^2}{\pi} \leq \frac{1}{4\pi} \left(\int_{S^2} f \right)^2 \leq \int_{S^2} f^2, \tag{13.6.1}$$

where equality in the second inequality occurs if and only if f is constant.

PROOF. **Step 1.** We use the fact that the Stiefel manifold M is the total space of a pair of Riemannian submersions p and q. We apply

Fubini's theorem twice to obtain

$$\begin{aligned}\int_{S^2} f &= \int_{S^2}\left(\frac{1}{2\pi}\int_{p^{-1}(v)} f\circ p\right)\\ &= \frac{1}{2\pi}\int_M f\circ p\\ &= \frac{1}{2\pi}\int_{S^2}\left(\int_{q^{-1}(w)} f\circ p\right)\\ &\geq \frac{1}{2\pi}\int_{S^2} m = 2m,\end{aligned}$$

proving the first inequality of (13.6.1).

Step 2. By the Cauchy–Schwarz inequality, we have

$$\left(\int_{S^2} 1\cdot f\right)^2 \leq 4\pi\int_{S^2} f^2,$$

proving the second inequality of (13.6.1).

Step 3. Equality occurs if and only if f and 1 are linearly dependent, i.e., if and only if f is constant. □

We define the quantity V_f by setting $V_f = \int_{S^2} f^2 - \frac{1}{4\pi}\big(\int_{S^2} f\big)^2$. Then Proposition 13.6.3 can be restated as follows.

Corollary 13.6.4. *Let $f\colon S^2 \to \mathbb{R}^+$ be continuous. Then*

$$\int_{S^2} f^2 - \frac{m^2}{\pi} \geq V_f \geq 0,$$

and $V_f = 0$ if and only if f is constant.

Proof. The proof is obtained from Proposition 13.6.3 by noting that $a \leq b \leq c$ if and only if $c - a \geq c - b \geq 0$. □

13.7. A probabilistic interpretation

We can assign a probabilistic meaning to the term V_f as follows. Let g_{can} be the canonical metric of curvature $K = 1$ on S^2.

Definition 13.7.1. Let μ be the probability measure induced by the metric $\frac{1}{4\pi}g_{can}$.

A function $f\colon S^2 \to \mathbb{R}^+$ is then thought of as a random variable with expectation $E_\mu(f) = \frac{1}{4\pi}\int_{S^2} f$. Its variance is thus given by

$$\mathrm{Var}_\mu(f) = E(f^2) - \big(E(f)\big)^2 = \frac{1}{4\pi}\int_{S^2} f^2 - \left(\frac{1}{4\pi}\int_{S^2} f\right)^2 = \frac{1}{4\pi}V_f.$$

The variance of a random variable f is non-negative, and it vanishes if and only if f is constant. This reproves the corresponding properties of V_f established above via the Cauchy–Schwarz inequality.

13.8. From the sphere to the real projective plane

Now let g_0 be the metric of constant Gaussian curvature $K = 1$ on $\mathbb{RP}^2$. The orientable double cover

$$\rho\colon (S^2, g_{can}) \to (\mathbb{RP}^2, g_0)$$

is a local isometry. Each projective line $C \subseteq \mathbb{RP}^2$ is the image under ρ of a great circle of S^2.

Proposition 13.8.1. *Given a function $f\colon \mathbb{RP}^2 \to \mathbb{R}^+$, we define $\bar{m} \in \mathbb{R}$ by setting*

$$\bar{m} = \min\left\{ \int_C f \colon C \subseteq \mathbb{RP}^2 \text{ a projective line} \right\}.$$

Then

$$\frac{2\bar{m}^2}{\pi} \;\le\; \frac{1}{2\pi}\left(\int_{\mathbb{RP}^2} f\right)^2 \;\le\; \int_{\mathbb{RP}^2} f^2,$$

where equality in the second inequality occurs if and only if f is constant.

Proof. We apply Proposition 13.6.3 to the composition $f\circ\rho$. Note that

$$\int_{\rho^{-1}(C)} f\circ\rho = 2\int_C f \quad \text{and} \quad \int_{S^2} f\circ\rho = 2\int_{\mathbb{RP}^2} f.$$

The condition for f to be constant holds since f is constant if and only if $f\circ\rho$ is constant. $\square$

For $\mathbb{RP}^2$ we define $\bar{V}_f = \int_{\mathbb{RP}^2} f^2 - \frac{1}{2\pi}\left(\int_{\mathbb{RP}^2} f\right)^2 = \frac{1}{2}V_{f\circ\rho}$. We obtain the following restatement of Proposition 13.8.1.

Corollary 13.8.2. *Let $f\colon \mathbb{RP}^2 \to \mathbb{R}^+$ be a continuous function. Then*

$$\int_{\mathbb{RP}^2} f^2 - \frac{2\bar{m}^2}{\pi} \ge \bar{V}_f \ge 0,$$

where $\bar{V}_f = 0$ if and only if f is constant.

Relative to the probability measure induced by $\frac{1}{2\pi}g_0$ on $\mathbb{RP}^2$, we have $E(f) = \frac{1}{2\pi}\int_{\mathbb{RP}^2} f$, and therefore $\mathrm{Var}(f) = \frac{1}{2\pi}\bar{V}_f$, providing a probabilistic meaning for the quantity $\bar{V}_f$, as before.

13.9. A generalisation of Pu's inequality

For a metric g_0 of constant Gaussian curvature $+1$ on S^2, we denote by dA_{g_0} the area element of the metric.

THEOREM 13.9.1. *Let g be a Riemannian metric on $\mathbb{RP}^2$. Let L be the shortest length of a noncontractible loop in $(\mathbb{RP}^2, g)$. Let $f\colon \mathbb{RP}^2 \to \mathbb{R}^+$ be such that $g = f^2 g_0$. Then*

$$\operatorname{area}(g) - \frac{2L^2}{\pi} \geq 2\pi \operatorname{Var}(f),$$

where the variance is with respect to the probability measure induced by $\frac{1}{2\pi} g_0$. Furthermore, equality $\operatorname{area}(g) = \frac{2L^2}{\pi}$ holds if and only if f is constant.

PROOF. **Step 1.** By the uniformization theorem, every metric g on $\mathbb{RP}^2$ is of the form $g = f^2 g_0$ where g_0 is a metric of constant Gaussian curvature $+1$ (unique up to isometry), and the function $f\colon \mathbb{RP}^2 \to \mathbb{R}^+$ is continuous.

Step 2. The area of g is $\int_{\mathbb{RP}^2} f^2 dA_{g_0}$. The g-length of a projective line C is $\int_C f$. Let L be the shortest length of a noncontractible loop. Then $L \leq \bar{m}$ where $\bar{m}$ is defined in Proposition 13.8.1, since a projective line in $\mathbb{RP}^2$ is a noncontractible loop.

Step 3. Corollary 13.8.2 implies $\operatorname{area}(\mathbb{RP}^2, g) - \frac{2L^2}{\pi} \geq \bar{V}_f \geq 0$.

Step 4. To characterize the boundary case of equality in Pu's inequality, note that if $\operatorname{area}(\mathbb{RP}^2, g) = \frac{2L^2}{\pi}$ then $\bar{V}_f = 0$, which implies that f is constant, by Corollary 13.8.2. Conversely, if f is a constant c, then the only geodesics are the projective lines, and therefore $L = c\pi$. Hence $\frac{2L^2}{\pi} = 2\pi c^2 = \operatorname{area}(\mathbb{RP}^2)$. $\square$

CHAPTER 14

Alternative proof of Pu's inequality

The alternative proof of Pu's inequality exploits certain properties of circle fibrations of the 3-sphere, and relies on the following ingredients:

(1) Hopf fibration;
(2) quaternions;
(3) geodesic flow[1] of a Riemannian manifold;
(4) a pair of orthogonal fibrations of the 3-sphere;
(5) a suitable integral-geometric identity.

14.1. Hopf fibration h

To prove Pu's inequality, we need to study the Hopf fibration of Section 4 more closely. The circle action in $\mathbb{C}^n$ restricts to the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$, which therefore admits a fixed-point-free circle action. Namely, the circle

$$S^1 = \{e^{i\theta} \colon \theta \in \mathbb{R}\} \subseteq \mathbb{C}$$

acts on a point $(z_1, \ldots, z_n) \in \mathbb{C}^n$ by

$$e^{i\theta}.(z_1, \ldots, z_n) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_n). \qquad (14.1.1)$$

Lemma 14.1.1. *The action is an isometry with respect to the natural Euclidean metric.*

Proof. In real coordinates, the matrix of this action by $e^{i\theta} = \cos\theta + i\sin\theta$ looks as follows for $n = 2$:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}$$

This is clearly an isometry. $\square$

The quotient manifold (space of orbits) S^{2n-1}/S^1 is the the complex projective space $\mathbb{CP}^{n-1}$. For $n = 2$ we get the 2-sphere S^2.

[1] Zrima geodesit

Definition 14.1.2. The quotient map

$$h\colon S^{2n-1} \to \mathbb{CP}^{n-1} \tag{14.1.2}$$

is called the *Hopf fibration*.

Definition 14.1.3. It will be convenient in the sequel to denote this data by a two-arrow diagram of the following type:

$$S^1 \longrightarrow S^{2n-1} \xrightarrow{h} \mathbb{CP}^{n-1}, \tag{14.1.3}$$

where the first arrow denotes the inclusion of a fiber, while the second arrow denotes the Hopf fibration h.

14.2. Hopf fibration is a Riemannian submersion

We specialize to the case $n = 2$.

Proposition 14.2.1. *Let S^3 be the unit 3-sphere. The Hopf fibration $h\colon S^3 \to S^2$ is a Riemannian submersion, for which the natural metric on the base S^2 is a metric of constant Gaussian curvature $+4$ and radius $\frac{1}{2}$.*

We note the following.

(1) The maximal distance between a pair of S^1 orbits is $\frac{\pi}{2}$ rather than π (see next item);
(2) a pair of antipodal points of S^3 lies in a common orbit and therefore descends to the same point of the quotient space S^2;
(3) a pair of points in S^2 at maximal distance is defined by the orbits of a pair $v, w \in S^3$ such that w is orthogonal to $\mathbb{C}v$;
(4) for such a pair we have $H(v, w) = 0$ where H is the Hermitian inner product in $\mathbb{C}^2$.

14.3. Hamilton quaternions

Definition 14.3.1. The algebra $\mathbb{H}$ of the Hamilton quaternions is the real 4-dimensional vector space with real basis $(1, i, j, k)$, so that

$$\mathbb{H} = \mathbb{R}\,1 + \mathbb{R}\,i + \mathbb{R}\,j + \mathbb{R}\,k$$

equipped with an associative,[2] distributive, and non-commutative product operation. This operation has the following properties:

(1) the center of $\mathbb{H}$ is $\mathbb{R}\,1$;
(2) the operation satisfies the relations $i^2 = j^2 = k^2 = -1$;
(3) also the relations $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

[2] chok kibutz

The relation of item (3) to the vector product in $\mathbb{R}^3$ is mentioned in Proposition 14.4.2.

LEMMA 14.3.2. *The algebra $\mathbb{H}$ has a natural structure of a complex vector space $\mathbb{C}^2$ via the identification $\mathbb{R}\,1 + \mathbb{R}\,i \simeq \mathbb{C}$.*

PROOF. The complementary subspace $\mathbb{R}\,j + \mathbb{R}\,k$ of $\mathbb{R}\,1 + \mathbb{R}\,i$ can be thought of as follows:

$$\mathbb{R}\,j + \mathbb{R}\,ij = (\mathbb{R}\,1 + \mathbb{R}\,i)j$$

using associativity and distributivity. We therefore obtain a natural isomorphism

$$\mathbb{H} \simeq \mathbb{C}1 + \mathbb{C}j,$$

showing that the pair of elements $(1, j)$ is a complex basis for $\mathbb{H}$. □

THEOREM 14.3.3. *Nonzero quaternions form a group under quaternionic multiplication.*

PROOF. Given $q = a + bi + cj + dk \in \mathbb{H}$, let $N(q) = a^2 + b^2 + c^2 + d^2$, and $\bar{q} = a - bi - cj - dk$. Then one checks that $q\bar{q} = N(q)$. Therefore q has a multiplicative inverse

$$q^{-1} = \frac{1}{N(q)}\bar{q},$$

proving the theorem. □

14.4. Complex structures on the algebra $\mathbb{H}$

DEFINITION 14.4.1. A *pure imaginary* Hamilton quaternion $q \in \mathbb{H}_0$ in $\mathbb{H} = \mathbb{R}\,1 + \mathbb{R}\,i + \mathbb{R}\,j + \mathbb{R}\,k$, is a real linear combination of i, j, and k.

Thus $\mathbb{H}_0 \subseteq \mathbb{H}$ is a real 3-dimensional subspace.

PROPOSITION 14.4.2. *With respect to the natural Euclidean metric on $\mathbb{H}_0$, quaternion multiplication of a pair of orthogonal elements coincides with the vector product on $\mathbb{R}^3$.*

PROOF. This is easily checked with respect to a standard basis such as i, j, k. □

More generally for pure quaternions p, q one has $p \times q = \frac{1}{2}(pq - qp)$ (see `https://en.wikipedia.org/wiki/Quaternion#Quaternions_and_the_space_geometry` for a more detailed discussion.)

COROLLARY 14.4.3. *Right or left multiplication by a unit quaternion q is an isometry of $\mathbb{H}$ equipped with the standard inner product of $\mathbb{R}^4$.*

Proof. The proof is analogous to that of Lemma 14.1.1, relying on Proposition 14.4.2. □

This can also be used using $N(qr) = N(q)N(r)$ which however also requires proof. Corollary 14.4.3 will be used in the proof of the following result, generalizing Lemma 14.3.2 on the natural complex structure on $\mathbb{H}$.

Proposition 14.4.4. *A choice of a pure imaginary quaternion $q \in \mathbb{H}_0$ specifies a natural complex structure*

$$\mathbb{H} = \mathbb{C}_q + \mathbb{C}_q^{\perp},$$

where $\mathbb{C}_q = \mathbb{R}\,1 + \mathbb{R}\,q$ and $\mathbb{C}_q^{\perp}$ is its orthogonal complement in $\mathbb{H}$.

Proof. Let $q = bi + cj + dk$ with $b, c, d \in \mathbb{R}$. We can assume without loss of generality that $|q| = 1$.

Step 1. By anticommutation among i, j, k, the cross-terms cancel out and we obtain

$$\begin{aligned} q^2 &= b^2i^2 + c^2j^2 + d^2k^2 + bc(ij + ji) + bd(ik + ki) + cd(jk + kj) \\ &= b^2i^2 + c^2j^2 + d^2k^2 \\ &= -1. \end{aligned}$$

Since $q^2 = -1$, the subspace $\mathbb{C}_q = \mathbb{R}\,1 + \mathbb{R}\,q$ with the product operation restricted from $\mathbb{H}$ is naturally isomorphic to the field $\mathbb{C}$.

Step 2. Let us show that the orthogonal complement of the subspace $\mathbb{C}_q \subseteq \mathbb{H} = \mathbb{R}^4$ has a natural structure of a complex line for multiplication by the "imaginary unit" q.

Let $r \in \mathbb{H}_0$ be a unit-norm quaternion orthogonal to $\mathbb{C}_q \subseteq \mathbb{H}$. We need to show that the quaternion qr is also orthogonal to $\mathbb{C}_q$. We multiply both sides of the dot product $qr \cdot 1$ by $q^{-1} = -q$. By Corollary 14.4.3, we obtain

$$qr \cdot 1 = r \cdot (q^{-1}1) = -r \cdot q = 0$$

by hypothesis on r, and similarly $qr \cdot q = r \cdot 1 = 0$. Thus qr is orthogonal to $\mathbb{C}_q$. Hence the orthogonal complement of $\mathbb{C}q$ is the subspace $\mathbb{C}_q r$, and we obtain the required orthogonal decomposition $\mathbb{H} = \mathbb{C}_q + \mathbb{C}_q r$. □

14.5. From complex structure to fibration

Recall that we have a decomposition $\mathbb{H} = \mathbb{R} + \mathbb{H}_0$ where

(1) $\mathbb{R} = \mathbb{R}\,1$ is the center of $\mathbb{H}$, and
(2) $\mathbb{H}_0$ is the space of pure imaginary quaternions.

$$\begin{tikzcd}
S^1 \arrow[dr] & & S^2 \\
& S^3 \arrow[ur, "f_r"] \arrow[dr, "f_q"] & \\
S^1 \arrow[ur] & & S^2
\end{tikzcd}$$

FIGURE 14.5.1. A pair of fibrations; cf. Figure 14.12.1

Consider the complex structure on $\mathbb{H} \simeq \mathbb{R}^4$ defined by a pure quaternion $q \in \mathbb{H}_0$, as in Proposition 14.4.4. As in formula (14.1.2), the choice of a complex structure leads to a Hopf fibration f_q of the unit 3-sphere $S^3 \subseteq \mathbb{H}$:

$$S^1 \longrightarrow S^3 \xrightarrow{f_q} S^2, \tag{14.5.1}$$

where each fiber of f_q is a circle $S^1 \subseteq S^3$ which is an orbit of the action by the unit circle in $\mathbb{C}_q$, similar to (14.1.1). Here we use the notation of Definition 14.1.3, where the first arrow in (14.5.1) denotes the inclusion of a fiber, while the second arrow denotes the Hopf map.

REMARK 14.5.1. Distinct pure imaginary quaternions $q, r \in \mathbb{H}_0$ define distinct complex structures and hence lead to distinct Hopf fibrations f_q, f_r.

Such a pair of fibrations, illustrated in diagram of Figure 14.5.1 following the notation used in (14.1.3), will play a crucial role in the proof of Pu's inequality in Section 14.13.

14.6. Lie groups

A Lie group is simultaneously a group and a smooth manifold, in such a way that the two structures are compatible. More precisely, we have the following definition.

DEFINITION 14.6.1. A *Lie group* is a manifold G with an associative[3] operation $\mu\colon G\times G \to G$ and inverse $\nu\colon G \to G$ with the following properties:

(1) the operation defines the structure of a group on G, so that in particular there exists an element $e \in G$ such that $\mu(e, x) = x$ for all $x \in G$, and furthermore $\mu(x, \nu(x)) = e$;
(2) both μ and ν are smooth maps.

[3] chok kibutz

COROLLARY 14.6.2. *The manifold S^3 admits the structure of a Lie group.*

PROOF. Nonzero quaternions form a group by Theorem 14.3.3. The multiplication restricts to the unit sphere $S^3 \subseteq \mathbb{R}^4 \simeq \mathbb{H}$ and defines a well-defined smooth product. The smooth inverse is given by the rule $q^{-1} = \bar{q}$ for each $q \in S^3$. $\square$

14.7. Lie group $SO(3)$ as quotient of S^3

As we mentioned in Section 14.6, the sphere S^3 can be thought of as the Lie group formed of the unit (Hamilton) quaternions in $\mathbb{H} \simeq \mathbb{C}^2$. In the sequel, an important role will be played by the Lie group $SO(3)$ of orthogonal 3×3 matrices of determinant 1. It turns out that the group

$$\mathrm{SO}(3, \mathbb{R}) = SO(\mathbb{H}_0)$$

can be identified with the quotient $S^3/\{\pm 1\}$ of the Lie group S^3 by its center, as follows.

Recall that conjugation by a quaternion is an isometry of $\mathbb{H}_0$.

PROPOSITION 14.7.1. *Consider $\mathbb{H}_0 = \mathbb{R}^3$.*

(1) *We have a natural isomorphism of Lie groups $\phi\colon S^3/\mathbb{Z}_2 \xrightarrow{\simeq} \mathrm{SO}(3, \mathbb{R})$, $\quad q \mapsto C_q$ where C_q is conjugation by q acting on $\mathbb{H}_0$.*
(2) *The nontrivial element in the kernel is the element $-1 \in S^3 \subseteq \mathbb{H}$.*

PROOF. Consider a unit quaternion $q \in S^3 \subseteq \mathbb{H}$. The homomorphism ϕ sends q to the isometry C_q of $\mathbb{H}_0 \simeq \mathbb{R}^3$ given by conjugation by q. Namely, $C_q\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the map $C_q(x) = q^{-1}xq$. Since the element -1 is in the center of $\mathbb{H}$, conjugation by -1 gives a trivial isometry $C_{-1} = Id$. Thus the homomorphism descends to the quotient $S^3/\{\pm 1\}$.

To get all the rotations in $\mathrm{SO}(3, \mathbb{R})$ around an axis $q_0 \in \mathbb{H}_0$, we conjugate by $\cos\theta\, 1 + \sin\theta\, q_0$, where $\theta \in \mathbb{R}$. $\square$

We obtain the following corollary.

COROLLARY 14.7.2. *The Hopf fibration f_q of S^3 defined by a pure quaternion $q \in \mathbb{H}_0$ descends to a fibration*

$$\mathrm{SO}(3, \mathbb{R}) \to S^2. \tag{14.7.1}$$

This result will be exploited in Section 14.9.

14.8. Unit sphere tangent bundle

Definition 14.8.1. Let M be a Riemannian manifold. The *unit sphere tangent bundle*, denoted T^uM, or the *unit tangent bundle* for short, is the submanifold of TM consisting of elements of unit norm:

$$T^uM = \big\{v \in TM \colon |v| = 1\big\}.$$

Proposition 14.8.2. *The choice of a unit tangent vector $v \in T^uS^n$ provides a natural identification F_v of the group $SO(n+1)$ with the unit tangent bundle of $M = S^n$.*

Proof. Choose a fixed unit tangent vector $v \in T^u(S^n)$. We will construct a natural identification

$$F_v \colon SO(n+1) \to T^u(S^n), \quad g \mapsto dg(v).$$

Namely, the group $SO(n+1)$ acts naturally on the sphere $S^n \subseteq \mathbb{R}^{n+1}$ by matrix multiplication. To construct the required identification F, we send an element $g \in SO(n+1)$, acting on S^n, to the image of the vector v under the tangent map (see Section 13.2) $dg \colon TS^n \to TS^n$. Namely, $F_v(g) = dg(v) \in T^u(S^n)$. $\square$

Corollary 14.8.3. *We have the following three distinct ways of viewing the same Lie group:*

(1) *the group* $\mathrm{SO}(3,\mathbb{R})$ *of orthogonal matrices;*
(2) *the quotient* $S^3/\{\pm 1\}$ *of the* 3*-sphere;*
(3) *the unit tangent bundle of the* 2*-sphere.*

The underlying smooth manifold can in fact be identified with the real projective space $\mathbb{RP}^3$, as well, as is obvious from item (2).

14.9. 2-sphere as a homogeneous space

The circle $\mathrm{SO}(2,\mathbb{R})$ of rotations of the (x,y)-plane can be viewed as the subgroup of $\mathrm{SO}(3,\mathbb{R})$ stabilizing the north pole, as follows.

Lemma 14.9.1. *The stabilizer*[4] *of the north pole $(0,0,1) \in S^2$ is the subgroup of matrices of the form*

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is isomorphic to $SO(2)$.

Let $H \subseteq G$ be a Lie subgroup of G.

[4] meyatzev

Definition 14.9.2. A *homogeneous space* G/H is the space of orbits of a Lie group G under the action by a subgroup H.

Corollary 14.9.3. *The sphere* S^2 *is the homogeneous space of the Lie group* $SO(3)$, *cf.* (14.7.1), *so that*

$$S^2 = SO(3)/SO(2). \tag{14.9.1}$$

See e.g., [**Ar83**, p. 82].

Definition 14.9.4. Given a point $\sigma \in S^2$ viewed as a homogeneous space via (14.9.1), we will denote by $SO(2)_\sigma$ the fiber over (i.e., the stabilizer of) σ.

Proposition 14.9.5. *The projection*

$$p\colon SO(3) \to S^2 \tag{14.9.2}$$

is a Riemannian submersion in both of the following cases:

(1) *for a metric of constant Gaussian curvature* $+4$ *on* S^2 *and the metric on* $SO(3)$ *is of constant sectional curvature* 1 *(i.e., antipodal quotient of the unit* 3*-sphere);*
(2) *for a metric of curvature* 1 *on the base* S^2 *and metric of curvature* $\frac{1}{4}$ *on the total space* $SO(3)$.

Remark 14.9.6. For the purposes of proving Pu's inequality, we will view fibration (14.9.2) as a fibration of the unit circle tangent bundle of the sphere:

$$p\colon T^uS^2 \to S^2 \tag{14.9.3}$$

using the identification of T^uS^2 and $SO(3)$ discussed in Section 14.7.

14.10. Geodesic flow on the tangent bundle

Let M be an n-dimensional Riemannian manifold. We define a geodesic in a coordinate patch as follows.

Definition 14.10.1. A geodesic $\alpha(t) = (\alpha^1(t), \cdots, \alpha^n(t))$ in M with initial point $p = (p^1, \ldots, p^n)$ and initial velocity $v = (v^1, \ldots, v^n)$ is defined using the Γ symbols of the Riemannian metric as a curve satisfying the ordinary differential equations

$$\alpha^{k''} + \Gamma^k_{ij}\, \alpha^{i'} \alpha^{j'} = 0, \quad k = 1, \ldots, n,$$

with the initial conditions $\alpha(0) = p$ and $\alpha'(0) = v$.

Definition 14.10.2. The geodesic starting at point $p \in M$ with initial velocity $v \in T_pM$ is denoted $\gamma(p, v, t)$.

It is easy to show[5] that a geodesic has constant speed. Therefore we can define the geodesic flow on the unit tangent bundle as follows.

DEFINITION 14.10.3. The *geodesic flow* on T^uM is the map

$$\mathbb{R} \times T^uM \to T^uM, \quad (t,(p,v)) \mapsto \big(\gamma(p,v,t), \gamma'(p,v,t)\big).$$

It was proved in differentialit 1 that great circles are geodesics on the sphere. Therefore we have the following.

LEMMA 14.10.4. *When $M = S^2$ with the standard metric normalized to have $K = 1$, the flow is periodic with period 2π, so that $\gamma(p,v,t+2\pi) = \gamma(p,v,t)$ when $|v| = 1$.*

Recalling that the unit tangent circle bundle of S^2 can be identified with $SO(3)$, we obtain the following theorem.

THEOREM 14.10.5. *Consider the geodesic flow of $M = S^2$. Then*
(1) *the flow defines a circle action $S^1 \times SO(3) \to SO(3)$;*
(2) *its homogeneous space (i.e., space of orbits) $\mathcal{D}$ is the space of oriented[6] great circles of S^2.*

Thus we have a fibration $\pi\colon SO(3) \to \mathcal{D}$.

COROLLARY 14.10.6. *The unit tangent bundle T^uS^2 admits the following pair of circle fibrations p and π:*
(1) *the fibration p over the base manifold S^2, with typical fiber $SO(2)_\sigma$ over each point $\sigma \in S^2$;*
(2) *the fibration π over the space $\mathcal{D}$ of orbits of the geodesic flow, where the typical fiber $\nu \subseteq T^uS^2$ is parametrized by the closed geodesic $\big(\gamma(p,v,t), \gamma'(p,v,t)\big)$.*

14.11. Dual real projective plane and its double cover

Let $\mathbb{RP}^2$ be the real projective plane. The orientable double cover of $\mathbb{RP}^2$ can be identified with S^2.

DEFINITION 14.11.1. Let $\mathbb{RP}^{2*}$ the dual real projective plane, namely the space of projective lines $\mathbb{RP}^1$ of $\mathbb{RP}^2$.

DEFINITION 14.11.2. Denote by $\mathcal{D}$ the orientable double cover of $\mathbb{RP}^{2*}$, identifiable with the space of oriented great circles (i.e., with the 2-sphere).

Here “D” is an allusion to both “double” and “dual.”

[5]See differentsialit 1 at https://u.cs.biu.ac.il/~katzmik/88-201.html
[6]mekuvanim

REMARK 14.11.3. The following observations should be kept in mind.

(1) The preimage of a real projective line under the double cover $S^2 \to \mathbb{RP}^2$ is a great circle on the 2-sphere;
(2) We will avoid using the notation S^2 for $\mathcal{D}$ so as not to confuse the base manifolds of two distinct fibrations p and π of Corollary 14.10.6;
(3) We think of $\mathcal{D}$ as the configuration space of oriented great circles on the 2-sphere.

DEFINITION 14.11.4. Let $\nu \in \mathcal{D}$ represent a generic oriented great circle. We will denote by $d\nu$ the measure (i.e., the area 2-form) on $\mathcal{D}$.

THEOREM 14.11.5. *The unit tangent bundle $T^u S^2$ can be represented as a subset of the Cartesian product $S^2 \times \mathcal{D}$ as follows:*

$$T^u S^2 = \left\{(x, \nu) \in S^2 \times \mathcal{D} \colon x \in \nu\right\}$$

PROOF. An oriented (directed) line through a point $x \in S^2$ defines a unique unit tangent vector at x, and vice versa. □

14.12. A pair of fibrations

We consider again the fibration $p\colon T^u S^2 \to S^2$ of formula (14.9.3). From the point of view of Theorem 14.11.5, this fibration sends (x, ν) to x:

$$p(x, \nu) = x.$$

By Theorem 14.11.5, there is a second fibration π of $T^u S^2$, defined by the formula $\pi(x, \nu) = \nu$. Thus, the total space $T^u S^2$ admits another Riemannian submersion, denoted π, over the configuration space of oriented circles:

$$\pi\colon T^u S^2 \to \mathcal{D}, \tag{14.12.1}$$

whose typical fiber ν is an orbit of the geodesic flow on $T^u S^2$ (see Section 14.10).

LEMMA 14.12.1. *Each orbit $\nu \subseteq T^u S^2$ of the geodesic flow projects under p to a great circle on the sphere.*

PROOF. A fiber of fibration π is the collection of unit vectors tangent to a given directed closed geodesic, i.e., great circle, on S^2. This great circle is precisely the image of ν under the projection p. □

REMARK 14.12.2. While fibration π may seem very different from fibration p, the two are actually equivalent from the quaternionic point of view; *cf.* Sections 14.3 and 14.5.

The diagram of Figure 14.12.1 illustrates the maps defined so far.

$$\begin{tikzcd}
SO(2) \arrow[dr] & & (\mathcal{D}, d\nu) \\
& T^uS^2 \arrow[ur, "\pi"] \arrow[dr, "p"] & \\
\nu \arrow[ur] \arrow[rr, "\text{great circle}"'] & & (S^2, d\sigma)
\end{tikzcd}$$

FIGURE 14.12.1. Integral geometry on the sphere; cf. Figure 14.5.1, (14.9.2) and (14.12.1)

14.13. Double fibration of T^uS^2, integral geometry on S^2

In this section, we prove Pu's inequality using integral geometry. The latter has its origin in results of P. Funk [**Fu16**] determining a symmetric function on the two-sphere from its great circle integrals; see [**Hel99**, Proposition 2.2, p. 59], as well as Preface therein.

Let $\mathbf{g}_0$ be the standard metric of constant Gaussian curvature $+1$ on S^2. Consider a metric $\mathbf{g} = f^2\mathbf{g}_0$, where $f > 0$ is a function on the 2-sphere.

THEOREM 14.13.1. *We have the following inequality:*

$$\text{area}(S^2, \mathbf{g}) \geq \frac{1}{\pi} L_{\min}^2, \tag{14.13.1}$$

where $L_{\min}$ is the least $\mathbf{g}$-length of a great circle of S^2. In the boundary case of equality, the function f must be constant.

PROOF. Let $d\sigma$ be the area element of $\mathbf{g}_0$. Applying the Cauchy–Schwarz inequality and Fubini's theorem twice (to Riemannian submersions p and π), we obtain[7]

$$\begin{aligned}
\operatorname{area}(S^2, \mathbf{g}) &= \iint_{S^2} f^2 d\sigma \\
&\geq \frac{1}{4\pi}\left(\iint_{S^2} f d\sigma\right)^2 \quad \text{(by Cauchy–Schwarz)} \\
&= \frac{1}{4\pi}\left(\frac{1}{2\pi}\iiint_{T^u S^2} f \circ p \ d\mathrm{vol}\right)^2 \quad \text{(by Fubini applied to } p\text{)} \\
&= \frac{1}{16\pi^3}\left(\iiint_{T^u S^2} f \circ p \ d\mathrm{vol}\right)^2 \\
&= \frac{1}{16\pi^3}\left(\iint_{\mathcal{D}} d\nu \left(\int_\nu f(t)dt\right)\right)^2 \quad \text{(by Fubini applied to } \pi\text{)} \\
&= \frac{1}{16\pi^3}\left(\iint_{\mathcal{D}} \operatorname{length}_{\mathbf{g}}(\nu) d\nu\right)^2 \\
&\geq \frac{1}{16\pi^3}\left(4\pi L_{\min}\right)^2 \\
&= \frac{1}{\pi} L_{\min}^2
\end{aligned}$$

Based on inequality (14.13.1), one proves Pu's inequality for the real projective plane as in Chapter 13. □

PROOF OF PU'S INEQUALITY. By the uniformisation theorem for the sphere S^2, every metric on S^2 is conformal to the standard one. A metric $\mathbf{g}$ on $\mathbb{RP}^2$ lifts to a centrally symmetric metric $\tilde{\mathbf{g}} = f^2 \mathbf{g}_0$ on S^2 whose conformal factor f is invariant under the antipodal map. Applying Theorem 14.13.1 to the metric $\tilde{\mathbf{g}}$, we obtain a great circle of $\tilde{\mathbf{g}}$-length L satisfying $A \geq \frac{1}{\pi} L^2$, where $A = \operatorname{area}(S^2, \tilde{\mathbf{g}})$. Note that $\operatorname{sys}_1(\mathbf{g}) \leq \frac{L}{2}$ and $\operatorname{area}(\mathbf{g}) = \frac{A}{2}$. Hence

$$\operatorname{sys}_1^2(\mathbf{g}) \leq \frac{L^2}{4} \leq \frac{\pi A}{4} = \frac{\pi}{2} \operatorname{area}(\mathbf{g}),$$

proving Pu's inequality. See [**Iv02**] for an alternative proof. □

14.13.1. Pu's inequality with isosystolic remainder term. This material is optional as it was already covered in Chapter 13. Consider the unit-area metric $\mathbf{g}_1 = \frac{1}{4\pi}\mathbf{g}_0$ on S^2, where $\mathbf{g}_0$ is the metric of Gaussian curvature $K = 1$. Thus, $\mathbf{g}_1$ is isometric to a 2-sphere of radius $\frac{1}{\sqrt{4\pi}}$ and

[7] This part could be modified to incorporate the variance remainder term.

constant Gaussian curvature $K = 4\pi$. Then

$$\mathbf{g} = f^2\mathbf{g}_0 = 4\pi f^2\mathbf{g}_1 = \left(\sqrt{4\pi}f\right)^2 \mathbf{g}_1.$$

Then the area form of $\mathbf{g}_1$ defines a probability measure μ. We will denote by $d\sigma$ the area form of $\mathbf{g}_0$.

LEMMA 14.13.2. *The integral $\int_{S^2} f\,d\sigma$ can be expressed in terms of the expected value of the random variable $\sqrt{4\pi}f$ by the formula*

$$\int_{S^2} f\,d\sigma = \sqrt{4\pi}\;E_\mu\left(\sqrt{4\pi}f\right).$$

PROOF. We have $E_\mu(\sqrt{4\pi}f) = \int_{S^2}\sqrt{4\pi}f\,d\,\mathrm{area}_{\mathbf{g}_1} = \int_{S^2}\sqrt{4\pi}f\frac{1}{4\pi}d\,\mathrm{area}_{\mathbf{g}_0} = \frac{1}{\sqrt{4\pi}}\int_{S^2} f\,d\,\mathrm{area}_{\mathbf{g}_0}$, proving the lemma. $\square$

LEMMA 14.13.3. *The integral $\int_{S^2} f^2\,d\sigma$ can be expressed in terms of the expected value of the random variable $(\sqrt{4\pi}f)^2 = 4\pi f^2$ by the formula*

$$\int_{S^2} f^2\,d\sigma = E_\mu\left(4\pi f^2\right).$$

PROOF. We have $E_\mu(4\pi f^2) = \int_{S^2} 4\pi f^2\,d\,\mathrm{area}_{\mathbf{g}_1} = \int_{S^2} 4\pi f^2\frac{1}{4\pi}d\,\mathrm{area}_{\mathbf{g}_0} = \int_{S^2} f^2\,d\,\mathrm{area}_{\mathbf{g}_0}$, proving the lemma. $\square$

LEMMA 14.13.4. *We have the following identity:*

$$\iint_{S^2} f^2 d\sigma - \frac{1}{4\pi}\left(\iint_{S^2} f d\sigma\right)^2 = Var_\mu\left(\sqrt{4\pi}f\right)$$

PROOF. The computational formula for the variance of the random variable $\sqrt{4\pi}f$ gives $E_\mu\left(4\pi f^2\right) - \left(E_\mu(\sqrt{4\pi}f)\right)^2 = \mathrm{Var}_\mu\left(\sqrt{4\pi}f\right)$. $\square$

COROLLARY 14.13.5. *Let $\mathbf{g} = f^2\mathbf{g}_0$. Then we have the following strengthened version of inequality* (14.13.1)*:* $\mathrm{area}(S^2,\mathbf{g}) - \frac{1}{\pi}L^2_{\min} \geq Var_\mu\left(\sqrt{4\pi}f\right)$.

COROLLARY 14.13.6. *We have the following strengthened form of Pu's inequality:*

$$\mathrm{area}(\mathbb{RP}^2,\mathbf{g}) - \frac{2}{\pi}\,\mathrm{sys}_1^2(\mathbb{RP}^2,\mathbf{g}) \geq \frac{1}{2}\,Var_\mu\left(\sqrt{4\pi}f\right) \tag{14.13.2}$$

where $\sqrt{4\pi}f$ is viewed as a random variable on the double cover of $\mathbb{RP}^2$.

An immediate application is the characterisation of the boundary case of equality in Pu's inequality. Namely if equality is attained in Pu's inequality then (14.13.2) implies $\mathrm{Var}_\mu\left(\sqrt{4\pi}f\right) = 0$ and therefore f is constant.

CHAPTER 15

Gromov's inequality for essential manifolds

15.1. Essential manifolds

We deal with Gromov's systolic inequality for essential manifolds.

To give some examples, all surfaces are essential with the exception of the sphere S^2. Real projective spaces $\mathbb{RP}^n$ are essential. If a manifold M admits a metric of negative curvature, then M is essential.

More generally, to define the property of being *essential* for an n-dimensional closed manifold M, we need a minimum of homology theory.

15.2. Gromov's inequality for essential manifolds

One of the deepest results in the field of systolic geometry is Gromov's inequality for the homotopy 1-systole of an essential n-manifold M:

$$\operatorname{sys}(M)^n \leq C_n \operatorname{vol}(M),$$

where C_n is a universal constant only depending on the dimension of M. Here the systole $\operatorname{sys}(M)$ is by definition the least length of a noncontractible loop in M.

Example 15.2.1. The inequality holds for all nonsimply connected surfaces, for real projective spaces, for all manifolds admitting a hyperbolic metric.

The proof involves a new invariant called the filling radius, introduced by Gromov, defined as follows.

15.3. Filling radius of a loop in the plane

The filling radius of a simple loop C in the plane is defined as the largest radius, $R > 0$, of a circle that fits inside C, see Figure 15.2.1:

$$\operatorname{Fillrad}(C \subseteq \mathbb{R}^2) = R.$$

There is a kind of dual point of view that allows one to generalize this notion in a fruitful way, as shown in [**Gro83**]. Namely, we consider the ϵ-neighborhoods of the loop C, denoted $U_\epsilon C \subseteq \mathbb{R}^2$, see Figure 15.3.1.

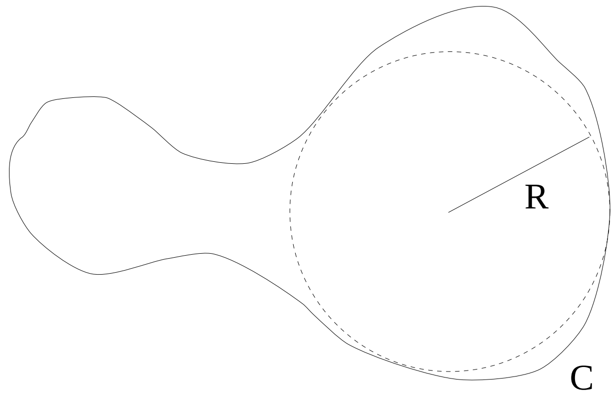

FIGURE 15.2.1. Largest inscribed circle has radius R

FIGURE 15.3.1. Neighborhoods of loop C

As $\epsilon > 0$ increases, the ϵ-neighborhood $U_\epsilon C$ swallows up more and more of the interior of the loop. The "last" point to be swallowed up is precisely the center of a largest inscribed circle. Therefore we can reformulate the above definition by setting

$$\mathrm{Fillrad}(C \subseteq \mathbb{R}^2) = \inf \{ \epsilon > 0 \,|\, \text{loop } C \text{ contracts to a point in } U_\epsilon C \} \,.$$

To define an *absolute* filling radius in a situation where M is equipped with a Riemannian metric g, Gromov exploits an embedding due to C. Kuratowski.

15.4. The sup-norm

On the space $L^\infty(X)$ of bounded Borel functions f on a set X, one can consider the norm

$$\|f\| = \sup_{x\in X} |f(x)|.$$

This norm is called the sup-norm.

15.5. Riemannian manifolds as spaces with a distance function

We consider a manifold M equipped with a distance function $d(x, y)$, where $x, y \in M$. In particular, a Riemannian metric on M defines such a distance function, by minimizing the length of all paths joining x to y.

15.6. Kuratowski embedding

One embeds M in the Banach space $L^\infty(X)$ of bounded Borel functions on M, equipped with the sup norm $\| \;\|$. Namely, we map a point $x \in M$ to the function $f_x \in L^\infty(M)$ defined by the formula $f_x(y) = d(x, y)$ for all $y \in M$, where d is the distance function defined by the metric. By the triangle inequality we have $d(x, y) = \|f_x - f_y\|$, and therefore the embedding is strongly isometric, in the precise sense that internal distance and ambient distance coincide. Such a strongly isometric embedding is impossible if the ambient space is a Hilbert space, even when M is the Riemannian circle (the distance between opposite points must be π, not 2!). We then set $E = L^\infty(M)$ in the formula above.

15.7. Homology theory for groups

Recall that the n-dimensional homology group is nontrivial. If M is orientable, then $H_n(M; \mathbb{Z}) = \mathbb{Z}$. If M is a non-orientable manifold, then $H_n(N, \mathbb{Z}_2) = \mathbb{Z}_2$. In either case, a generator of the homology group is denoted $[M]$ and called the *fundamental class.*

There is a parallel homology theory for groups. In particular, one can consider the homology of the fundamental group $\pi = \pi_1(M)$, namely $H_n(\pi)$. One additional ingredient we need to know is the existence of a natural homomorphism

$$\phi\colon H_n(M) \to H_n(\pi).$$

Then M is called essential if its fundamental class $[M]$ maps to a nonzero class in the homology of its fundamental group:

$$\phi([M]) \neq 0 \in H_n(\pi).$$

15.8. Relative filling radius

We first define a notion of a filling radius of M relative to an embedding. Given an embedding of M in Euclidean space E, let $\epsilon > 0$, and denote by $U_\epsilon M$ the neighborhood in E consisting of all points at distance at most ϵ from M. Let

$$\phi_\epsilon \colon H_n(M) \to H_n(E)$$

be the inclusion homomorphism induced by the inclusion of M in its ϵ-neighborhood $U_\epsilon M$ in E. We set

$$\mathrm{Fillrad}(M \subseteq E) = \inf \left\{ \epsilon > 0 \,|\, \phi_\epsilon([M]) = 0 \in H_n(U_\epsilon M) \right\},$$

15.9. Absolute filling radius

We define the filling radius of M to be its relative filling radius relative to the canonical embedding in $L^\infty(M)$, by setting

$$\mathrm{FillRad}(M) = \mathrm{FillRad}\left(M \subseteq L^\infty(M)\right).$$

Namely, Gromov proved a sharp inequality relating the systole and the filling radius: $\mathrm{sys}\pi_1 \leq 6\ \mathrm{FillRad}(M)$, valid for all essential manifolds M; as well as an inequality

$$\mathrm{FillRad} \leq C_n \mathrm{vol}_n{}^{\frac{1}{n}}(M),$$

valid for all closed manifolds M.

A summary of a proof, based on recent results in geometric measure theory by S. Wenger, building upon earlier work by L. Ambrosio and B. Kirchheim, appears in Section 12.2 of the book "Systolic geometry and topology" referenced below.

15.10. Systolic freedom

First examples of systolic freedom were given by Gromov in [**Gro96**, pp. 350–351]. The systolic freedom of $S^n \times S^n$ for $n \geq 3$ was proved by Katz in ([**10**], 1995). The systolic freedom of $S^2 \times S^2$ was proved by Katz and Suciu in ([**12**], 1999). For other results see [**BaK98**], [**Fr99**].

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