88-826 DIFFERENTIAL GEOMETRY
HOMEWORK SET 2

1. Let K be a field, let V be a vector space over K, and let $\Lambda(V)$ be its exterior algebra. Thus for any 1-form $v \in \Lambda(V)$, we have $v \wedge v = 0$. Prove that if the characteristic (me'afyen) of K is different from two, then $v \wedge w = -w \wedge v$ for all 1-forms $v, w \in \Lambda(V)$.

2. Prove that every decomposable (simple) 2-form η on $\mathbb{R}^4$ satisfies $\eta \wedge \eta = 0$.

3. Let $A \in \Lambda^2(\mathbb{R}^4)$ be defined by the formula

$$A = e_1 \wedge e_2 + e_3 \wedge e_4. \tag{0.1}$$

Prove that $A \wedge A \neq 0$, and conclude that A is not decomposable (simple).

4. Thinking of the symplectic form A on $\mathbb{R}^4$ as the imaginary part of a Hermitian inner product, prove that the comass norm of A equals 1.

5. Consider the standard flag (degel) in $\mathbb{C}^4$, and consider the corresponding decomposition of $\mathbb{CP}^3$ into cells (ta'im). Let e^4 be the 4-dimensional cell of the decomposition. Prove that its closure in $\mathbb{CP}^3$ is a copy of $\mathbb{CP}^2$.

6. On the unit circle S^1, consider the standard 1-form traditionally denoted $d\theta$. Prove that $d\theta$ is not a coboundary, i.e. it is not in the image of the differential $d : C^\infty(S^1) \to \Omega^1(S^1)$. (Hint: use Stokes' theorem.)

7. Show that if two elements α, β of the k-th homology group of M differ by a element of finite order, then α and β have the same stable norm.

8. Let M be an orientable four-dimensional manifold with $b_2(M) = 1$, with an integer de Rham class $\omega \in L^2_{\mathrm{dR}}(M)$ such that the cup-square ω^2 is a generator of $L^4_{\mathrm{dR}}(M)$. Prove that every Riemannian metric g on M satisfies the stable systolic inequality $\mathrm{stsys}_2(g)^2 \leq 2\,\mathrm{vol}(g)$.

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