July 4, 2022

DIFFERENTIAL GEOMETRY 88-826 HOMEWORK SET 5

**Due Date: 22 june '22**

1. Let $r > 0$ and let $D$ be the unbounded region

$$D = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \geq r^2\}$$

endowed with the standard orientation $dx \wedge dy$. Determine the induced orientation on $\partial D$ and compare it to $d\theta$.

2. Let $M$ be a 6-dimensional manifold with $b_2(M) = 1$, with an integer de Rham class $\omega \in L^2_{\mathrm{dR}}(M)$ such that $\omega^{\cup 3}$ is the fundamental cohomology class of $M$. Prove that every Riemannian metric $g$ on $M$ satisfies the stable systolic inequality $\mathrm{stsys}_2(g)^3 \leq 6\mathrm{vol}(g)$.

3. Let $M$ be the Cartesian product of the manifolds $\mathbb{CP}^1$, $\mathbb{CP}^2$, ..., $\mathbb{CP}^n$. Prove that all metrics $g$ of unit volume (i.e., volume 1) on $M$ satisfy $\mathrm{stsys}_2(g) \leq C_n$ for a suitable constant $C_n$ independent of the metric.

4. Determine which of the following 8-dimensional manifolds satisfy a stable systolic inequality for $\mathrm{stsys}_2$ with a constant independent of the metric:

(1) $S^2 \times S^8$;
(2) $S^2 \times \mathbb{CP}^3$;
(3) $S^2 \times S^2 \times S^4$;
(4) $S^2 \times S^2 \times \mathbb{CP}^2$;
(5) $\mathbb{CP}^2 \times S^4$.