March 9, 2014 Differential geometry 88-826 Homework 1

1. Consider the curve $\alpha(t) = (2\cos t, 2\sin t)$ in the (u, v)-plane. Consider the derivation X on the space $\mathbb{D}_p$ of smooth functions $f \in \mathbb{D}_p$ near the point $p = (\sqrt{2}, \sqrt{2})$ given by $X(f) = \frac{d}{dt}(f(\alpha(t))|_{t=\pi/4}$. Express X as a linear combination of the partial derivatives $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ at the point p.

2. The volume of an open region $D \subset \mathbb{R}^3$ is calculated with respect to cylindrical coordinates (r, θ, z) using the volume element

$$dV = r\,dr\,d\theta\,dz.$$

Namely, an integral is of the form $\int_D dV = \iiint r\,dr\,d\theta\,dz$.

(a) Find the volume of a right circular cone with height h and base a circle of radius b.

(b) evaluate the integral $\iiint\limits_E \sqrt{x^2+y^2}zdV$ where E is the cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 2$.

(c) Find the volume of the object filling the region above the paraboloid $z = x^2 + y^2$ and below the plane $z = 1$.

3. Spherical coordinates (ρ, θ, ϕ) range between the bounds $0 \leq \rho$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$ (note the different upper bounds for θ and ϕ). The area of a spherical region D is calculated using a volume element of the form $dV = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$, so that the volume of a region D is $\int_D dV = \iiint\limits_D \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$.

(1) Find the volume of the region above the cone $\phi = \beta$ and inside the sphere of radius $\rho = c$.

(2) Find the integral $\iiint\limits_E x^2+y^2+z^2dV$, where E is the solid region bounded by the sphere $x^2 + y^2 + z^2 = b^2$.

(3) Find the integral $\iiint\limits_E \frac{1}{x^2+y^2+z^2}dV$, where E is the region between two spheres: $a \leq \rho \leq b$.

4. Let $\delta^i_{\ j}$ be the Kronecker delta function on $\mathbb{R}^n$, where $i, j = 1, \ldots, n$, viewed as a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$. Evaluate the expression

$$\delta^i_{\ j}\delta^j_{\ k}\delta^k_{\ i}.$$