88826 Differential geometry 2015 targil 3

1. This problem deals with flows on manifolds.
   (a) Define the notion of a flow $\theta(t,p)$ on a manifold M.
   (b) Define the notion of an infinitesimal generator X of the flow θ.
   (c) Prove that a vector field on a manifold is invariant under its flow θ_t.

2. Let $f:\mathbb{R}\to\mathbb{R}$ be a real function.
   (a) State the definition of the *microcontinuity* of a function *at a point*, and express the property of *continuity of f on its domain* D_f assuming $D_f=\mathbb{R}$, in terms of microcontinuity.
   (b) Express the property of uniform continuity of f on its domain D_f, assuming $D_f=\mathbb{R}$, in terms of microcontinuity.
   (c) Analyze the behavior of $f(x)=x^2$ in terms of microcontinuity and uniform continuity.
   (d) Given a continuous function f on $[0,1]$, define a hyperfinite partition and use it to prove the extreme value theorem for f.

3. Let $\mathbb{C}$ be the field of complex numbers. Let $A\subset\mathbb{C}$ be the subfield consisting of all points of the form $a+ib$ where $a,b\in\mathbb{Q}$.
   (a) Give a detailed definition of the structure ${}^*\mathbb{Q}$ of hyperrational numbers in terms of a nonprincipal ultrafilter $\mathcal{F}\subset\mathcal{P}(\mathbb{N})$.
   (b) Determine whether ${}^*\mathbb{Q}$ contains nonzero infinitesimals, and if so provide an example.
   (c) Let ${}^*A=\{a+ib:\ a,b\in{}^*\mathbb{Q}\}$. Let $I=\{a+ib\in{}^*A:\ \mathrm{st}(a)=0,\ \mathrm{st}(b)=0\}$. Let $B=\{a+ib\in{}^*A:\ a \text{ is finite},\ b \text{ is finite}\}$. Determine whether the quotient B/I is isomorphic to either of the structures $\mathbb{N},\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C},A$ or their natural extensions ${}^*\mathbb{N}$, ${}^*\mathbb{Z},{}^*\mathbb{Q},{}^*\mathbb{R}$, ${}^*\mathbb{C}$, ${}^*A$.

4. Let $\mathcal{F}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and ${}^*\mathbb{R}$ the corresponding hyperreal line.
   (a) Consider a sequence $\mathcal{A}=\langle A_n\subset\mathbb{R}:n\in\mathbb{N}\rangle$ of subsets of $\mathbb{R}$. Give a detailed definition of the internal subset $[\mathcal{A}]\subset{}^*\mathbb{R}$.
   (b) Specify when a hyperreal $u=[u_n]\in{}^*\mathbb{R}$ belongs to $[\mathcal{A}]$.
   (c) Consider the sequence $\langle 1,2,3,\ldots\rangle$ and let H be the corresponding hyperreal. Determine whether the set $[0,H]=\{x\in{}^*\mathbb{R}:0\le x\le H\}$ is internal, and if so describe it by a sequence of sets as in part (a).
   (d) Determine whether the set $\{x\in{}^*\mathbb{R}:0\le x\le\pi\}$ is internal, and if so describe it by a sequence of sets as in part (a).