June 25, 2026

DIFFERENTIAL GEOMETRY 88-826 HOMEWORK SET 4

Due Date: 1 july '26

1. Let $r > 0$ and let D be the *un*bounded region

$$D = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \geq r^2\}$$

endowed with the standard orientation $dx \wedge dy$. Determine the induced orientation on ∂D and compare it to $d\theta$.

2. Let M be a 6-dimensional manifold with $b_2(M) = 1$, with an integer de Rham class $\omega \in L^2_{\mathrm{dR}}(M)$ such that $\omega^{\cup 3}$ is the fundamental cohomology class of M. Find the best possible bound for all Riemannian metrics g on M of the ratio $\frac{\mathrm{stsys}_2(g)}{\sqrt[3]{\mathrm{vol}(g)}}$, with proof.

3. Let $M = \mathbb{CP}^1 \times \mathbb{CP}^2 \times \mathbb{CP}^3$. Prove that all metrics g of volume 1 on M satisfy $\mathrm{stsys}_2(g) \leq C_n$ for a suitable constant C_n independent of the metric.

4. Determine which of the following 8-dimensional manifolds satisfy a stable systolic inequality relating stsys_2 and the volume, with a constant independent of the metric:

(1) $S^2 \times S^6$;
(2) $S^2 \times \mathbb{CP}^3$;
(3) $S^2 \times S^2 \times S^4$;
(4) $S^2 \times S^2 \times \mathbb{CP}^2$;
(5) $\mathbb{CP}^2 \times S^4$;
(6) $\mathbb{CP}^3 \times T^2$.