March 24, 2015

Differential geometry 88-826-01 homework set 2

1. Let $\mathbb{R}$ act on the manifold $M = \mathbb{R}^2$ by means of the flow $\theta_t(x, y)$ acting according to the formulas

$$x \mapsto x \cos t + y \sin t, \quad y \mapsto -x \sin t + y \cos t,$$

i.e., $\theta_t(x, y) = (x \cos t + y \sin t, -x \sin t + y \cos t)$.

(a) Show that this is a globally defined action of $\mathbb{R}$ on M.
(b) find the infinitesimal generator X of this flow.
(c) Describe the orbits of this flow.

2. Consider the action of $\mathbb{R}$ on $M = \mathbb{R}^2$ given by the flow $\theta_t(x, y) = (xe^{2t}, ye^{-3t})$.

(a) Show that this is a C^∞ action.
(b) Determine the infinitesimal generator X.
(c) Show that the infinitesimal generator is θ-invariant.

3. Consider the manifold $M = \mathrm{GL}(2, \mathbb{R})$ and define an action of $\mathbb{R}$ on M by the formula

$$\theta(t, A) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} A$$

via matrix multiplication of $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and A for all $A \in \mathrm{GL}(2, \mathbb{R})$. Find the infinitesimal generator X of this action.

4. If c is an upper bound for a set $A \subset \mathbb{R}$ we will write $A \le c$. The completeness property of $\mathbb{R}$ asserts that if A is bounded from above, then there is a least upper bound $d \in \mathbb{R}$ for A, or in formulas

$$(\forall A \subset \mathbb{R})\big[(\exists c \in \mathbb{R})[A \le c] \rightarrow (\exists d \in \mathbb{R})[A \le d] \wedge (\forall e \in \mathbb{R})[A \le e \rightarrow d \le e]\big] \tag{1}$$

(a) Express the condition $A \le c$ by a explicit first-order formula with quantification only over numbers.
(b) Reformulate the completeness property (1) in a way amenable to an application of the transfer principle as explained in class.
(c) Apply the transfer principle to the resulting formula so as to obtain a correct statement over ${}^*\mathbb{R}$.
(d) Give an example of the failure of the naive application of transfer to (1).