Differential geometry 88-826 homework set 2

Due Date: 19 april '23

1. Let A and B be copies of $\mathbb{R}^3$, with coordinates $u = (u^1, \ldots, u^2)$ in A and $v = (v^1, \ldots, v^2)$ in B, and with transition function $u = \phi(v) = \frac{v}{v \cdot v}$ whenever $v \in \mathbb{R}^3 \setminus \{0\}$. Prove that the resulting manifold M with coordinate patches A and B is metrizable.

2. Let $M = T^3$ be the 3-torus. Prove that the tangent bundle of M is diffeomorphic to the product $T^3 \times \mathbb{R}^3$.

3. Recall that the m-th exterior power $\bigwedge^m(\mathbb{R}^m)$ of $\mathbb{R}^m$ is spanned by the single element $\omega = e_1 \wedge e_2 \wedge \cdots \wedge e_m$. Consider the 2-multivector

$$\alpha = e_1 \wedge e_2 + e_3 \wedge e_4 + \cdots + e_{2n-1} \wedge e_{2n} \in \bigwedge\nolimits^2(\mathbb{R}^{2n}).$$

Express the product $\alpha \wedge \alpha \wedge \cdots \wedge \alpha$ (n times) explicitly as a multiple of $\omega \in \mathbb{R}^{2n}$.

4. Let M be a n-dimensional Riemannian manifold. Consider a coordinate chart (A, u) in M. Let f be a smooth function on A and consider the differential 2-form $\eta = f(u^1, \ldots, u^n) du \wedge dv$ in A, where du and dv are among the coordinate forms du^i. Prove that the 4-form $dd\eta$ identically vanishes.