Differential geometry 88-826 homework set 2

Due Date: 10 jun '26

1. Recall that the m-th exterior power $\bigwedge^m(\mathbb{R}^m)$ of $\mathbb{R}^m$ is spanned by the single element $\omega = e_1 \wedge e_2 \wedge \cdots \wedge e_m$. Consider the 2-multivector

$$\alpha = e_1 \wedge e_2 + e_3 \wedge e_4 + \cdots + e_{2n-1} \wedge e_{2n} \in \bigwedge^2(\mathbb{R}^{2n}).$$

Express the product $\alpha \wedge \alpha \wedge \cdots \wedge \alpha$ (n times) explicitly as a multiple of $\omega \in \mathbb{R}^{2n}$.

2. Let M be a n-dimensional Riemannian manifold. Consider a coordinate chart (A, u) in M. Let f be a smooth function on A and consider the differential 2-form

$$\eta = f(u^1, \ldots, u^n) du \wedge dv$$

in A, where du and dv are among the coordinate forms du^i. Prove that the 4-form $dd\eta$ identically vanishes.

3. Compute the Euclidean norm $|\alpha|$ and the comass $\|\alpha\|$ of the 2-form $\alpha = e_1 \wedge e_2 + e_1 \wedge e_3 + \cdots + e_1 \wedge e_n$ on $\mathbb{R}^n$.

4. Consider the Eisenstein lattice $L_E \subseteq \mathbb{C}$ spanned by the cube roots of unity. Let L_E^* be its dual lattice. Calculate the product $\lambda_1(L_E^*)\, \lambda_1(L_E)$.

5. Let $M = \{z \in \mathbb{C} : (|z|^2 - 1)(|z-3|^2 - 1) = 0\}$. Compute the de Rham cohomology groups $H^0_{dR}(M)$ and $H^1_{dR}(M)$.