88-826 DIFFERENTIAL GEOMETRY, MOED A, 21 AUG '08

Duration of the exam: $2\frac{1}{2}$ hours.

All answers must be justified by providing complete proofs.

1. Let $a, b > 0$ and assume $a < b$. Consider the ellipsoid $\mathcal{E}_{a,a,b} \subset \mathbb{R}^3$ with half-axes a, a, b given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

(a) Prove that $\mathcal{E}_{a,a,b}$ is centrally symmetric.
(b) Let $\mathcal{P} = \mathcal{E}_{a,a,b}/\pm 1$ be the real projective plane obtained as the antipodal quotient of the ellipsoid defined in part (a). Calculate the systole $\operatorname{sys}_1(\mathcal{P})$ of $\mathcal{P}$.
(c) Define the notion of the Riemannian diameter of a manifold M.
(d) Calculate the Riemannian diameter of $\mathcal{P}$.

2. For each of the following lattices L, find L^* and compute $\lambda_1(L^*)$, after presenting the definition in part (a):

(a) Define the notion of the dual lattice in Euclidean n-space.
(b) The lattice $L_G \subset \mathbb{C}$ spanned over $\mathbb{Z}$ by the roots of $z^4 = 16$.
(c) Let $a, b, c > 0$ such that $a \le b \le c$. The lattice $L_{a,b,c} \subset \mathbb{R}^3$ is spanned by ae_1, be_2, and ce_3.
(d) The lattice $L_E \subset \mathbb{C}$ spanned by the roots of $z^6 = 64$.

3. Let M be an closed connected orientable 6-dimensional manifold. Assume that $b_2(M) = 1$.

(a) Define what it means for a de Rham class $\omega \in H^2_{\mathrm{dR}}(M)$ to be an integer class.
(b) Assume $\omega \in L^2_{\mathrm{dR}}(M)$ is an integer de Rham class such that the class $\omega \cup \omega \cup \omega$ is a generator of $L^6_{\mathrm{dR}}(M)$. Evaluate the expression $\int_M \omega \cup \omega \cup \omega$.
(c) Given a metric g on M, define the comass norms $\| \;\|$ in $\Lambda^2(T^*_p M)$ and $\| \;\|_\infty$ in $\Omega^2 M$.
(d) Let $\eta \in \omega$ be a representative differential form. Estimate the integral $\int_M \eta \wedge \eta \wedge \eta$ in terms of the comass as well as the total volume $\operatorname{vol}(M)$ of M.
(e) Find the best upper bound for the ratio $\frac{\operatorname{stsys}_2(g)^3}{\operatorname{vol}(g)}$.

4. Consider the polar coordinates (r, θ) of a point p in the Euclidean plane.

(a) Find a natural orthonormal basis, in terms of the polar coordinates, for the cotangent plane T_p^* at p when p is not the origin.
(b) Find a natural orthonormal basis, in terms of the polar coordinates, for the tangent plane T_p when p is not the origin.
(c) Consider the cotangent line $T_p^* S^1$ at a point p of the circle of radius $r_0 > 0$. Consider the lattice $L_0 \subset T_p^*$ spanned by the 1-form $d\theta$. Calculate $\lambda_1(L_0)$.
(d) Consider the tangent line T_p at a point p of the circle of radius $r_0 > 0$. Consider the lattice $L_1 \subset T_p$ spanned by $\frac{\partial}{\partial \theta}$. Calculate $\lambda_1(L_1)$.
(e) Determine whether or not the differential form $d\theta$ on S^1 is a coboundary, i.e. lies in the image of the map $C^\infty(S^1) \to \Omega^1(S^1)$ defined by the exterior derivative.

5. Let M be a Riemannian manifold.

(a) Define the stable norm $\| \ \|$ in the homology group $H_k(M)$ of M.
(b) Prove that if two elements α, β differ by an element of finite order, then their stable norms coincide.
(c) Define the invariants $\mathrm{sys}\pi_1$ and stsys_1.
(d) Explain why $\mathrm{sys}\pi_1$ and stsys_1 coincide in the case of a Riemannian torus $M = \mathbb{T}^2$.

GOOD LUCK!

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