88826 Differential geometry, moed A, 1 aug '16

Duration of the exam: 3 hours. Each problem is worth 22 points.

All answers must be justified by providing complete proofs.

1. This problem deals with flows on manifolds.
   (a) Define the notion of a flow $\theta(t,p)$ on a manifold M.
   (b) Define the notion of an infinitesimal generator X of the flow θ.
   (c) Prove that a vector field on a manifold is invariant under its flow θ_t.

2. Let $f : D_f \to \mathbb{R}$ be a real function with domain D_f, and f^* its natural extension.
   (a) Formulate the definition of microcontinuity of a function *at a point.*
   (b) express the property of *continuity of f on its domain D_f* assuming $D_f = \mathbb{R}$, in terms of microcontinuity.
   (c) Express the property of uniform continuity of f on its domain D_f, assuming $D_f = \mathbb{R}$, in terms of microcontinuity, and analyze the behavior of $f(x) = x^2$ in terms of microcontinuity and uniform continuity.
   (d) Given a continuous real function f on $[0,1] \subseteq \mathbb{R}$, define a hyperfinite partition and use it to prove the extreme value theorem for f.

3. Let $\mathbb{C}$ be the field of complex numbers. Let $A \subset \mathbb{C}$ be the subfield consisting of all points of the form $a + ib$ where $a, b \in \mathbb{Q}$.
   (a) Give a detailed definition of $\mathbb{Q}^*$ (hyperrational numbers) in terms of a nonprincipal ultrafilter $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$.
   (b) Determine whether $\mathbb{Q}^*$ contains nonzero infinitesimals, and if so provide an example.
   (c) Let $A^* = \{a+ib : \ a, b \in \mathbb{Q}^*\}$. Let $I = \{a+ib \in A^* : \ \mathrm{st}(a) = 0, \ \mathrm{st}(b) = 0\}$. Let $B = \{a + ib \in A^* : \ \ a \text{ is finite}, \ b \text{ is finite}\}$. Determine whether the quotient B/I is isomorphic to either of the structures $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, A$ or their natural extensions $\mathbb{N}^*$, $\mathbb{Z}^*, \mathbb{Q}^*, \mathbb{R}^*$, $\mathbb{C}^*$, A^*, with proof.

4. Let $\mathcal{F}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and $\mathbb{R}^*$ the corresponding hyperreal line.
   (a) Consider a sequence $A = \langle A_n \subset \mathbb{R} : n \in \mathbb{N}\rangle$ of subsets of $\mathbb{R}$. Give a detailed definition of the internal subset $\mathcal{A} = [A] \subset \mathbb{R}^*$.
   (b) Specify when a hyperreal $u = [u_n] \in \mathbb{R}^*$ is said to belong to $\mathcal{A}$.
   (c) Consider the sequence $\langle 1, 2, 3, \ldots\rangle$ and let H be the corresponding hyperreal. Determine whether the set $\{x \in \mathbb{R}^* : 0 \le x \le H\}$ is internal, and if so describe it by a sequence of sets as in part (a).
   (d) Determine whether the set $\{x \in \mathbb{R}^* : 0 \le x \le \sqrt{2}\}$ is internal, and if so describe it by a sequence of sets as in part (a).

5. This problem deals with small oscillations of a pendulum.
   (a) Give the equation of a pendulum of length ℓ under gravitational acceleration g.
   (b) Present the problem as a first order system.
   (c) Show that by an appropriate choice of infinitesimal λ, oscillations with amplitude on the order of λ are all periodic with period independent of the amplitude, and specify the period.

GOOD LUCK!