88-826 Differential Geometry, moed A

Bar Ilan University, Prof. Katz, Date: 11 aug '17

Duration of the exam: 3 hours. Each problem is worth 20 points.

All answers must be justified by providing complete proofs.

1. Let H be a positive infinite hyperreal. Determine if the following functions are microcontinuous at H:
   (a) $f(x) = \sin x$;
   (b) $g(x) = \sin(x^2)$;
   (c) $h(x) = x^2$;
   (d) $j(x) = \sqrt{x}$.

2. Let ${}^*\mathbb{N} = \mathbb{N}^{\mathbb{N}}/\mathcal{F}$ where $\mathcal{F}$ is a free ultrafilter. Elements of ${}^*\mathbb{N}$ are called hypernaturals. A hypernatural is called infinite if it is greater than every element of the subset $\mathbb{N} \subseteq {}^*\mathbb{N}$, and finite otherwise.
   (a) Prove that every finite hypernatural is a natural number.
   (b) Prove that the ordered set $({}^*\mathbb{N}, <)$ is not well-ordered.

3. Use a hyperfinite partition to prove the intermediate value theorem: If f is a continuous real function on the unit interval $[0, 1]$ such that $f(0) \cdot f(1) < 0$ then there exists a point $c \in [0, 1]$ such that $f(c) = 0$.

4. Let $\langle A_n \colon n \in \mathbb{N}\rangle$ be a nested sequence of nonempty sets of real numbers.
   (a) prove that the sequence $\langle {}^*A_n \colon n \in \mathbb{N}\rangle$ has a common point.
   (b) We will say that a set is compact if any countable cover by open sets has a finite subcover. Prove that a set A is compact if and only if every point of ${}^*A$ is nearstandard.

5. Let Φ and G be D^1 prevector fields.
   (a) Prove that $\Phi \circ G$ and $G \circ \Phi$ are equivalent prevector fields.
   (b) Prove that $a \mapsto a + \delta_\Phi + \delta_G$ is also a D^1 prevector field.

GOOD LUCK!