**88-826 Differential Geometry, moed A**
**Bar Ilan University, Prof. Katz**
**Date: 24 jul '18**
Duration of the exam: 3 hours
Each of 5 problems is worth 20 points; bonus problem is 10 points
**All answers must be justified by providing complete proofs**

1. In parts (a)-(c), does there exist a real constant $C$ such that the following relation holds for all infinitesimal $\varepsilon$, and if so which $C$?
   (a) $\cos(1+\varepsilon) \sqcap\!\!\sqcap 1 + C\varepsilon^2$;
   (b) $e^{\varepsilon} \sqcap\!\!\sqcap 1 + C\varepsilon$;
   (c) $\ln(1-\varepsilon) \sqcap\!\!\sqcap C\varepsilon$.
   (d) Let $H$ be an infinite number. Determine the order of magnitude (seder godel) of the expression $\sqrt{H^2-1} - \sqrt{H^2-4}$.

2. Let $A$ be a field and let ${}^*\!A = A^{\mathbb{N}}/\mathcal{F}$ where $\mathcal{F}$ is a free ultrafilter. Let $A_F \subseteq {}^*\!A$ be the subring of finite elements, and let $A_I$ be the subring of infinitesimal elements. Identify the quotient $A_F/A_I$ in each of the following cases:
   (a) $A = \mathbb{Q}$;
   (b) $A = \mathbb{R}$;
   (c) $A = \mathbb{C}$.

3. The extreme value theorem (EVT) states that if $f$ is a continuous real function on the unit interval $[0,1]$ then $f$ has a maximum.
   (a) Apply a hyperfinite partition using an infinite integer $H$ to prove the EVT;
   (b) use the EVT to prove Rolle's theorem: a differentiable function on a compact interval with identical values at the endpoints has vanishing derivative at some interior point of the interval;
   (c) use Rolle's theorem to prove the mean value theorem: if $f$ is a differentiable function then $(\forall x \in \mathbb{R})(\forall h \in \mathbb{R})(\exists \vartheta \in \mathbb{R}) \big[f(x+h) - f(x) = h \cdot g(x+\vartheta h)\big]$ where $0 < \vartheta < 1$ and $g(x) = f'(x)$.

4. Let $\langle A_n \colon n \in \mathbb{N}\rangle$ be a decreasing nested sequence of nonempty sets of real numbers: $A_n \subseteq \mathbb{R}$.
   (a) prove that the sequence $\langle {}^*\!A_n \colon n \in \mathbb{N}\rangle$ has a common point.
   (b) We will say that a set $S \subseteq \mathbb{R}$ is compact if each countable cover of $S$ by open sets has a finite subcover. Prove that $S$ is compact if and only if every point of ${}^*S$ is nearstandard.

5. Let $\Phi$ and $G$ be $D^1$ prevector fields on $\mathbb{R}^n$ generated respectively by displacements $\delta_\Phi$ and $\delta_G$.

(a) Prove that $\Phi \circ G$ and $G \circ \Phi$ are equivalent prevector fields.
(b) Prove that $a \mapsto a + \delta_\Phi + \delta_G$ is also a $D^1$ prevector field.

**Bonus question**. Let $H$ be an infinite hyperreal. Prove that the function $f(x) = \sin(x^2)$ is not microcontinuous at $H$.

GOOD LUCK!