88-826 Differential geometry, moed A, 20 july '11

Duration: $2\frac{1}{2}$ hours. **Justify all answers and provide complete proofs.**

1. Given a metric g on a torus $\mathbb{T}^2$, let $\lambda_1 = \lambda_1(\mathbb{T}^2, g)$ be the length of a shortest noncontractible loop (lul'ah bilti-kvitzah) $\gamma_0 \subset \mathbb{T}^2$. Let $\tau = \tau(\mathbb{T}^2, g)$ be the conformal parameter of the torus. Let $a > b > 0$, and consider the 2-parameter family $g_{a,b}$ of tori of revolution in 3-space (with circular section) obtained by rotating the circle $(x-a)^2 + z^2 = b^2$ around the z-axis.
   (a) Calculate the conformal parameter $\tau(g_{a,b})$.
   (b) Calculate $\lambda_1(g_{a,b})$ in terms of the parameters a, b.
   (c) Give the definition of the first homology group $H_1(\mathbb{T}^2; \mathbb{Z})$.
   (d) Let λ_2 the least length of a noncontractible loop whose homology class is not proportional to that of the loop γ_0 as above. Calculate λ_2 in terms of the parameters a, b.

2. Let $D \subset \mathbb{C}$ be the unit disk. Let $S^1 = \partial D$ its boundary circle. Let $E \subset \mathbb{C}$ be the complement of the interior of D.
   (a) Given a 2-form η on D and a vector $v \in T_pD$, define the interior product operation $v \lrcorner \eta$. Explain how to induce an orientation from a domain to its boundary.
   (b) Consider the standard orientation $dx \wedge dy = r dr d\theta$ in $\mathbb{C}$, and its restriction to $D \subset \mathbb{C}$. Describe explicitly the induced orientation on S^1.
   (c) Consider the standard orientation $dx \wedge dy = r dr d\theta$ in $\mathbb{C}$, and its restriction to $E \subset \mathbb{C}$. Describe explicitly the induced orientation on S^1.
   (d) Compare the orientations on S^1 resulting from (b) and (c).

3. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. Let $H^k_{\mathrm{dR}}(\mathbb{T}^2)$ be its de Rham cohomology group.
   (a) Exploit the exterior differential complex to calculate $H^0_{\mathrm{dR}}(\mathbb{T}^2)$ and $H^3_{\mathrm{dR}}(\mathbb{T}^2)$.
   (b) Investigate the following hypothesis: when does a $\mathbb{Z}^2$-periodic function $f(x,y)$ with zero mean (i.e., one has the following: $\int_0^1 \int_0^1 f(x,y)dxdy = 0$) define an exact 2-form?
   (c) Exploit the function $a(x) = \int_0^1 f(x,t)dt$ to determine when an arbitrary 2-form defined by a function with zero mean, is the exterior derivative of a suitable 1-form $gdx + hdy$.
   (d) Use the information obtained in (b) and (c) so as to calculate $H^2_{\mathrm{dR}}(T^2)$.

4. Let $\mathbb{C}^\nu$ be the complex vector space.
   (a) Define the symplectic form A on $\mathbb{C}^\nu$, and calculate A^μ.
   (b) State and prove Wirtinger's inequality for the 3rd power power A^3 of A.

5. Let M be an closed connected orientable 8-dimensional manifold. Assume that $b_2(M) = 1$ and that for an $\omega \in H^2_{dR}(M)$ one has $\omega^{\cup 4} \neq 0$.
   (a) Define what it means for a de Rham class $\omega \in H^2_{\mathrm{dR}}(M)$ to be an integer class.
   (b) Given a metric g on M, define the norm $\| \;\|$ in $\Lambda^2(T^*_pM)$; the norm $\| \;\|_\infty$ in $\Omega^2 M$; and the norm $\| \;\|^*$ in de Rham cohomology.
   (c) Let $\eta \in \omega$ be a representative 2-form. Estimate the integral $\int_M \eta \wedge \eta \wedge \eta \wedge \eta$ in terms of the comass of η as well as the total volume $\mathrm{vol}(M)$ of M.
   (d) Find the best upper bound for the ratio $\mathrm{stsys}_2(g)^4 / \mathrm{vol}(g)$.