88826 Differential geom., moed B, 15 sep '14

Duration of the exam: 3 hours.

All answers must be justified by providing complete proofs.

1. Let D_p be the space of functions $f(x, y, z)$ differentiable in a neighborhood of a point $p \in \mathbb{R}^3$.
   (a) Give a detailed definition of the notion of a 1-form on D_p.
   (b) Give a detailed definition of the notion of a derivation at the point p.
   (c) Calculate, with proof, the dimension of the space of derivations at the point $p = 0 \in \mathbb{R}^3$.

2. Let C be the curve in the (x, z)-plane defined by the equation $(x - 3)^2 + z^2 = 4$. Let $M \subset \mathbb{R}^3$ be the surface of revolution obtained by rotating C around the z-axis, with parametrisation $\underline{x}(\theta, \phi)$.
   (a) Find the coefficients of the first and second fundamental forms and the Gaussian curvature $K(\theta, \phi)$ of M.
   (b) Determine when the Gaussian curvature is positive, first as a function of θ, ϕ, and then also in terms of the coordinates (x, y, z).
   (c) Find the lattice $L \subset \mathbb{R}^2$ corresponding to the conformal class of the metric on M.
   (d) Calculate the corresponding parameter τ in the standard fundamental domain.

3. This problem deals with flows on manifolds.
   (a) Define the notion of a flow $\theta(t, p)$ on a manifold M.
   (b) Define the notion of an infinitesimal generator X of the flow θ.
   (c) Prove that a vector field on a manifold is invariant under its flow θ_t.

4. This problem deals with various notions related to continuity.
   (a) State the definition of microcontinuity of a function at a point, and determine whether the functions $x^{1/3}$ and x^3 are microcontinuous at 0 and at $H \in {}^*\mathbb{N} \setminus \mathbb{N}$.
   (b) Given a real function f on a domain $D_f \subset \mathbb{R}$, express the property of *continuity* of f in its domain when $D_f = \mathbb{R}$ in terms of microcontinuity.
   (c) Given a real function f on a domain $D_f \subset \mathbb{R}$, express the property of *uniform continuity* of f in its domain when $D_f = \mathbb{R}$ in terms of microcontinuity.

(d) Assume that a continuous function f on $[0,1]$ satisfies $f(0) < 0$ and $f(1) > 0$. Use a hyperfinite partition to give a detailed proof of the intermediate value theorem for f.

5. Let $\mathcal{F}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and ${}^*\mathbb{R}$ the corresponding hyperreal line.

(a) Present a detailed definition of ${}^*\mathbb{R}$ in terms of $\mathcal{F}$.

(b) Consider a sequence $\mathcal{A} = \langle A_n \subset \mathbb{R} : n \in \mathbb{N} \rangle$ of subsets of $\mathbb{R}$. Give a detailed definition of the internal subset $[\mathcal{A}] \subset {}^*\mathbb{R}$ and specify when a hyperreal $u = [u_n] \in {}^*\mathbb{R}$ belongs to $[\mathcal{A}]$.

(c) Consider the sequence $\langle 1, 2, 3, \ldots \rangle$ and let H be the corresponding hyperreal. Determine whether the set $[0, H] = \{x \in {}^*\mathbb{R} : 0 \leq x \leq H\}$ is internal, and if so describe it by a sequence of sets as in part (b).

(d) Determine whether the set $\{x \in {}^*\mathbb{R} : 0 \leq x \leq \pi\}$ is internal, and if so describe it by a sequence of sets as in part (b).

GOOD LUCK!