88826 Differential geometry, moed B, 1 sep '16

Duration of the exam: 3 hours.

All answers must be justified by providing complete proofs.

1. Let f be a real function.
   (a) Assume that f satisfies $f(x)-f(y) < (x-y)^2$ for all $x, y \in [0,1]$. Prove that f is necessarily constant (you may use a hyperfinite partition or any other method).
   (b) Let $c \in [0,1]$. Assume that f is differentiable at c and that $f'(c) > 0$. Prove that there is an $x > c$ in the domain such that $f(x) > f(c)$ (you may use the transfer principle or any other method).
   (c) Assume f is defined on $\mathbb{R}$ and satisfies $f(x+y) = f(x) + f(y)$ for all x, y. Assume furthermore that f is continuous at 0. Let $c \in \mathbb{R} \setminus \{0\}$. Prove that f is continuous at c.

2. If c is an upper bound for a set $A \subset \mathbb{R}$ we will write $A \le c$. The completeness property of $\mathbb{R}$ asserts that if A is bounded from above, then there is a least upper bound $d \in \mathbb{R}$ for A, or in formulas

$$(\forall A \subseteq \mathbb{R})$$
$$\big[(\exists c \in \mathbb{R})[A \le c] \Rightarrow (\exists d \in \mathbb{R})[A \le d] \wedge (\forall e \in \mathbb{R})[A \le e \Rightarrow d \le e]\big].$$

   (a) Express the condition $A \le c$ by an explicit first-order formula with quantification only over numbers.
   (b) Reformulate the completeness property given by the formula above in a way amenable to an application of the transfer principle.
   (c) Apply the transfer principle to the resulting formula so as to obtain a correct statement over ${}^*\mathbb{R}$.
   (d) Give an example of the failure of the naive application of transfer to the displayed formula above.

3. This problem deals with flows on manifolds.
   (a) Define the notion of a flow $\theta(t, p)$ on a manifold M.
   (b) Define the notion of an infinitesimal generator X of the flow θ.
   (c) Prove that a vector field on a manifold is invariant under its flow θ_t.
   (d) Let F a prevector field on a manifold. Define the corresponding hyperreal walk and prove that F is invariant under the hyperreal walk.

4. This problem deals with various notions related to continuity.

   (a) State the definition of microcontinuity of a function at a point, and determine whether the functions $x^{1/3}$ and x^3 are microcontinuous at 0 and at $H \in {}^*\mathbb{N} \setminus \mathbb{N}$.

   (b) Given a real function f on a domain $D_f \subset \mathbb{R}$, express the property of *continuity* of f in its domain when $D_f = \mathbb{R}$ in terms of microcontinuity.

   (c) Given a real function f on a domain $D_f \subset \mathbb{R}$, express the property of *uniform continuity* of f in its domain when $D_f = \mathbb{R}$ in terms of microcontinuity.

   (d) Assume that a continuous function f on $[0, 1]$ satisfies $f(0) < 0$ and $f(1) > 0$. Use a hyperfinite partition to give a detailed proof of the intermediate value theorem for f.

5. Let $\mathcal{F}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and ${}^*\mathbb{R}$ the corresponding hyperreal line.

   (a) Present a detailed definition of ${}^*\mathbb{R}$ in terms of $\mathcal{F}$.

   (b) Consider a sequence $\mathcal{A} = \langle A_n \subset \mathbb{R} : n \in \mathbb{N} \rangle$ of subsets of $\mathbb{R}$. Give a detailed definition of the internal subset $[\mathcal{A}] \subset {}^*\mathbb{R}$ and specify when a hyperreal $u = [u_n] \in {}^*\mathbb{R}$ belongs to $[\mathcal{A}]$.

   (c) Consider the sequence $\langle 1, 2, 3, \ldots \rangle$ and let H be the corresponding hyperreal. Determine whether the set $[0, H] = \{x \in {}^*\mathbb{R} : 0 \le x \le H\}$ is internal, and if so describe it by a sequence of sets as in part (b).

   (d) Let $\gamma \in \mathbb{R}$ be the Euler constant $\gamma = \lim_{n\to\infty} \left(-\ln n + \sum_{k=1}^{n} \frac{1}{k} \right)$. Determine whether the set $\{x \in {}^*\mathbb{R} : 0 \le x \le \gamma\}$ is internal, and if so describe it by a sequence of sets as in part (b).

GOOD LUCK!