88-826 Differential Geometry, moed B
Bar Ilan University, Prof. Katz
Date: 8 august '19 Duration of the exam: 3 hours. Each of 5 problems is worth 20 points; bonus problem is 10 points
All answers must be justified by providing complete proofs

1. Let $\mathbb{T}^n$ be the n-dimensional torus and let S^n be the n-dimensional sphere.
   (a) Specify an atlas for $\mathbb{T}^1$ and prove that $\mathbb{T}^1$ is a smooth manifold.
   (b) Specify an atlas for S^2 and prove that S^2 is a smooth manifold.

2. This problem concerns the exterior differential complex on a manifold M.
   (a) Give detailed definitions of the differentials d_1 and d_2 in the following segment of the exterior differential complex: $\Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \Omega^3(M)$.
   (b) Prove that the segment is exact, i.e., $d_2 \circ d_1(\xi) = 0$ for all 1-forms $\xi \in \Omega^1(M)$.

3. For each of the following lattices L, find L^* and compute $\lambda_1(L^*)$:
   (a) The lattice $L_G \subseteq \mathbb{C}$ spanned over $\mathbb{Z}$ by the roots of $z^4 = 81$.
   (b) Let $a, b, c > 0$ such that $a \le b \le c$. The lattice $L_{a,b,c} \subseteq \mathbb{R}^3$ is spanned by ae_1, be_2, and ce_3.
   (c) The lattice $L_E \subseteq \mathbb{C}$ spanned by the roots of $z^6 = 64$.

4. Let $\mathbb{T}^n$ be the n-dimensional torus.
   (a) Compute the de Rham cohomology group $H^0_{dR}(\mathbb{T}^n)$.
   (b) Compute the de Rham cohomology group $H^1_{dR}(\mathbb{T}^1)$.

5. Let M be a closed connected orientable 6-dim. manifold. Assume that $b_2(M) = 1$ and that for an $\omega \in H^2_{dR}(M)$ one has $\omega^{\cup 3} \neq 0$.
   (a) Given a metric g on M, provide a detailed definition of the comass norms $\| \ \|$ in $\Lambda^2(T_p^*M)$ and $\| \ \|_\infty$ in $\Omega^2 M$.
   (b) Let $\eta \in \omega$ be a representative differential form, where ω is a 2-dimensional generator for de Rham cohomology. Estimate the integral $\int_M \eta \wedge \eta \wedge \eta$ in terms of the comass as well as the total volume $\mathrm{vol}(M)$ of M.
   (c) Find the best upper bound for the ratio $\frac{\mathrm{stsys}_2(g)^3}{\mathrm{vol}(g)}$.

6. (**bonus**) Let $\mathbb{R}/\mathbb{Z}$ denote the circle of length 1. Consider the cylinder $C_H = \mathbb{R}/\mathbb{Z} \times [0, H]$ of height $H > 0$, with coordinates $x \in \mathbb{R}/\mathbb{Z}$ and $y \in [0, H]$. Suppose a surface M contains an annulus conformally equivalent to C_H. Find the best upper bound for the ratio $\frac{\mathrm{sys}_1^2(M)}{\mathrm{area}(M)}$.

Good luck!