October 17, 2010 Differential geometry 88-526 Homework 0

1. Given below are symmetric matrices. Which of the matrices are positive definite? Negative definite? Indefinite?

(a) $\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$; (b) $\begin{pmatrix} 3 & 4 \\ 4 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} -2 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{pmatrix}$; (d) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

2. For each of the following functions f, determine whether the given stationary point p corresponds to a local minimum, local maximum, or saddle point (nekudat ukaf).

(a) $f(x, y) = 3x^2 - xy + y^2$, $p = (0, 0)$;
(b) $f(x, y) = \sin x + y^3 + 3xy + 2x - 3y$, $p = (0, -1)$;
(c) $f(x, y) = x^3 + xyz + y^2 - 3x$, $p = (1, 0, 0)$.

3. The polar coordinates (r, θ) in the plane arise naturally in complex analysis (of one complex variable). They satisfy $r^2 = x^2 + y^2$ and $x = r\cos\theta$, $y = r\sin\theta$. It is shown in elementary calculus that the area of a region D in the plane in polar coordinates is calculated using the area element

$$dA = r\,dr\,d\theta.$$

Thus, an integral is of the form

$$\int_D dA = \iint r dr d\theta.$$

Find the area of the following regions:

(a) $0 \leq r \leq 3$; $-\pi/2 \leq \theta \leq \pi/2$;
(b) $2 \leq r \leq 4$; $0 \leq \theta \leq \pi/4$;
(c) $0 \leq \theta \leq \pi$; $0 \leq r \leq \theta$.

3. Cylindrical coordinates (r, θ, z) in Euclidean 3-space are studied in Vector Calculus. They are a natural extension of the polar coordinates in the plane. The volume of an open region D is calculated with respect to spherical coordinates using the volume element

$$dV = r\,dr\,d\theta\,dz.$$

Namely, an integral is of the form

$$\int_D dV = \iiint r dr d\theta\,dz.$$

(a) find the volume of a right circular cone with height h and base a circle of radius b.
(b) evaluate the integral $\iiint_E \sqrt{x^2 + y^2} z dV$ where E is the cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 2$.

(c) Find the volume of the object filling the region above the paraboloid $z = x^2 + y^2$ and below the plane $z = 1$.

4. Spherical coordinates

$$(\rho, \theta, \phi)$$

in Euclidean 3-space are studied in Vector Calculus. The coordinate ρ is the distance from the point to the origin, satisfying $\rho^2 = x^2 + y^2 + z^2$, or $\rho^2 = r^2 + z^2$, where $r^2 = x^2 + y^2$. If we project the point orthogonally to the (x, y)-plane, the polar coordinates of its image, (r, θ), satisfy $x = r\cos\theta$ and $y = r\sin\theta$. The last coordiate ϕ of the spherical coordinates is the angle between the position vector of the point and the third basis vector e_3 in 3-space (pointing upward along the z-axis). Here we have the bounds $0 \leq \rho$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$ (note the different upper bounds for θ and ϕ). Recall that the area of a spherical region D is calculated using a volume element of the form

$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi,$$

so that the volume of a region D is

$$\int_D dV = \iiint_D \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$$

(1) Find the volume of the region above the cone $\phi = \beta$ and inside the sphere of radius $\rho = c$.
(2) Find the integral $\iiint_E x^2 + y^2 + z^2 dV$, where E is the sphere $x^2 + y^2 + z^2 = b^2$.
(3) Find the integral $\iiint \frac{1}{x^1+y^2+z^2} dV$, where E is the region between two spheres: $a \leq \rho \leq b$.