20 oct '10 Differential geometry 88-526-01 homework set 1

1. Let a, b, c be vectors in $\mathbb{R}^3$. Let "$\times$" be the vector product and "$\cdot$", the scalar product.
   (a) Prove the identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$.
   (b) TBA.

2. Let $f(u^1, u^2) = (u^1)^3 + (u^2)^3$, and $\alpha(t) = (\cos t, \sin^2 t)$.
   (a) Evaluate $\frac{d}{dt}|_{t=\frac{\pi}{2}}(f \circ \alpha)$ by chain rule.
   (b) Evaluate $\frac{d}{dt}|_{t=\frac{\pi}{2}}(f \circ \alpha)$ by direct calculation.

3. In this exercise, we view matrices as linear transformations.
   (a) write down and prove the associative law of matrix multiplication in terms of Einstein summation convention;
   (b) write down and prove the distributive law of matrix multiplication in terms of Einstein summation convention.

4. Let $B = (b_{ij})$ where $b_{ij} = i^2 + 2j$, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n$.
   (a) Find the coefficients $b_{\{ij\}}$ of the symmetrisation S of B;
   (b) find the coefficients $b_{[ij]}$ of the antisymmetrisation A of B.

5. Let A and B be square matrices of the same size, viewed as linear transformations. Using Einstein summation convention, prove the following relation for traces: that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

6. Let B be the $n \times n$ matrix all of whose entries equal 1, viewed as a bilinear form Q_B. Let q be the quadratic form associated to Q_B. Let $\{e_i\}$ be the standard basis of $\mathbb{R}^n$. Calculate $q(v)$ where v is the vector $v = \sum_i ie_i$.

7. Let $\delta^i{}_j$ be the Kronecker delta function on $\mathbb{R}^n$, where $i, j = 1, \ldots, n$, viewed as a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$. Evaluate the expression

$$\delta^i{}_j \delta^j{}_k \delta^k{}_i.$$

8. Consider the function $f(x, y) = x^2 + 3xy + 2y^2 + x + 2y$.
   (a) Find the critical point of f;
   (b) determine its nature: maximum, minumum, or saddle point (nekudat 'ukaf).

9. Determine the nature of the critical point p for the given function:
   (a) $f(x, y) = 3x^2 - xy + y^2$, at $p = (0, 0)$;
   (b) $f(x, y) = \sin x + y^3 + 3xy + 2x - 3y$, at $p = (0, -1)$.