5 november 2006

Differential geometry 88-526-01 homework set 2

1. Recall that $\lambda_k(L)$ is the k-th successive minimum of a lattice L. Consider the lattice $L_\tau \subset \mathbb{C}$ defined by $L_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}1$. Let $\tau = t\sqrt{-1}$, where $t > 0$.

(a) Calculate $\lambda_1(L_\tau)$ as a function of t.
(b) Calculate $\lambda_2(L_\tau)$ as a function of t.
(c) Graph both functions (a) and (b).

2. Let $a, b, \omega \in \mathbb{R}^+$ be positive real numbers. Consider a helix $\alpha(t) = (a\cos\omega t, a\sin\omega t, bt)$.

(a) Find the quadratic equation of a cylinder containing the image of the helix.
(b) Parametrize the helix by arc length (unit speed parameter).
(c) Calculate the curvature (akmumiut) k_α of the helix in terms of the parameters a, b, ω.

3. Find the curvature of the curve of intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $x + y + z = 1$.

4. Let $\alpha(s)$ be a unit speed curve in $\mathbb{R}^2$, and assume its curvature is nonzero: $k_\alpha(s) \neq 0$. Let $T(s) = \alpha'(s)$ be its tangent vector.

(a) Let $m(s) = \tan\theta(s)$ be the slope of the tangent line at the point $\alpha(s) \in \mathbb{R}^2$. Express the components of $T(s)$ in terms of $\theta(s)$.
(b) Define the normal vector $N(s)$ by the formula $\alpha''(s) = k_\alpha(s)N(s)$. Find an orthogonal 2 by 2 matrix A such that $N(s) = A(T(s))$.

5. Find the total area of the region R_i on the pseudosphere

$$x(\theta, \phi) = \left(e^\phi \cos\theta, e^\phi \sin\theta, \int_\phi^0 \sqrt{1 - e^{2\psi}}\, d\psi \right),$$

where the region R_i is defined by

(a) $R_1 = \{-\pi \leq \phi \leq 0\}$;
(b) $R_2 = \{-\infty < \phi \leq 0\}$.

6. Let $g_{ij}(u^1, u^2)$ be the metric coefficients of a surface, while $\Gamma^k_{ij}(u^1, u^2)$, its Christoffel symbols. Prove that $\Gamma^k_{12} g_{k1} = \frac{1}{2} g_{11;2}$.

7. Consider a surface of revolution M parametrized by

$$x(\theta, \phi) = (f(\phi)\cos\theta, f(\phi)\sin\theta, g(\phi)).$$

(a) Calculate the Christoffel symbols Γ^2_{ij}.
(b) The general geodesic equation $(\alpha^2)'' + \Gamma^2_{ij}(\alpha^i)'(\alpha^j)' = 0$ on M can be written down explicitly in terms of f, g, θ, ϕ, and their derivatives. Write down the explicit form of the equation (the final equation must not contain any α's, but rather only f, g, θ, and ϕ).