December 13, 2010
Differential geometry   88-526   homework set 4

1. Calculate the curvature of the curve obtained as the intersection $M_1 \cap M_2$ of the following surfaces:

(a) surface $M_1 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 25\}$, while surface $M_2 = \{(x, y, z) \in \mathbb{R}^3 | z = 4\}$;

(b) surface $M_1 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 25\}$, while surface $M_2 = \{(x, y, z) \in \mathbb{R}^3 | (x-6)^2 + y^2 + z^2 = 25\}$.

2. Consider a regular surface $x(u^1, u^2)$ in $\mathbb{R}^3$.

(a) Define what is meant by the regularity of $x(u^1, u^2)$;

(b) prove that the expression $\frac{\partial}{\partial u^m}\left(\Gamma_{ij}^k x_k + L_{ij} n\right)$ is symmetric with respect to j and m;

(c) write the expression $L_{i[j} L_{k]}^q$ in terms of the Γ symbols alone.

3. Let $\rho > 0$ be a real number. Calculate the Gaussian curvature of the metric

$$\frac{1}{\left(1 + \frac{\rho}{4}(x^2 + y^2)\right)^2}(dx^2 + dy^2),$$

in other words the metric obtained from the standard flat metric in the plane by multiplying by the conformal factor $\lambda = f^2$ where

$$f(x, y) = \left(1 + \tfrac{\rho}{4}(x^2 + y^2)\right)^{-1}.$$

4. Let $z = \sqrt{4 - x^2}$ be a curve in the (x, z) plane.

(a) Find a parametrisation of the corresponding surface of revolution M in $\mathbb{R}^3$;

(b) calculate the mean curvature of M.

5. Let $x(u^1, u^2)$ be a regular surface in $\mathbb{R}^3$. Let $\beta = x \circ \alpha$ be a curve. Assume that for all $t > 0$, the vector $\beta''(t)$ is proportional to $x_1 \times x_2$. Find an ordinary differential equation satisfied by the components $\alpha^k(t)$ of α.