Differential geometry 88-526 homework set 5
January 13, 2011

1. Given a metric g on a torus, let $\tau \in D$ be its conformal parameter, where $D \subset \mathbb{C}$ is the standard fundamental domain

$$D = \{z = x + iy \in \mathbb{C} : |x| \le \tfrac{1}{2}, y > 0, |z| \ge 1\}.$$

Thus, g is conformally equivalent to a flat metric $\mathbb{C}/L$ where the lattice L is spanned by $\tau \in \mathbb{C}$ and $1 \in \mathbb{C}$, so that $L = \mathbb{Z}\tau + \mathbb{Z}1$. If g is a torus of revolution in $\mathbb{R}^3$, let λ_φ be the length of the φ-loop, let $\lambda_{\theta_{\min}}$ be the least length of a θ-loop, and let $\lambda_{\theta_{\max}}$ be the maximal length of a θ-loop (see Section 11.9 of the course notes). Consider the torus of revolution whose generating curve is the circle in the (x, z) plane centered at the point with coordinates $(3, 4)$ and of radius 2.

(a) Find λ_φ of the torus.
(b) Find $\lambda_{\theta_{\min}}$ of the torus.
(c) Find $\lambda_{\theta_{\max}}$ of the torus.
(d) Find the conformal parameter τ of the torus.

2. Let $x(u^1, u^2)$ be a parametrized surface in 3-space. Express the following quantities in terms of the coefficents g_{ij} of the first fundamental form; the symbols Γ^k_{ij}; the coefficents $L^i_{\ j}$ of the Weingarten map; and the coefficients L_{ij} of the second fundamental form, simplifying the expression as much as possible. Here the Einstein summation convention implies summation over every index occurring both in a lower position and in an upper position.

(a) $\langle x_{ij}, x_k \rangle g^{i\ell}$.
(b) $\langle n_i, x_k \rangle g^{ik}$.
(c) $\langle n_i, x_j \rangle g^{i\ell}$.
(d) $\langle x_{ij}, n_k \rangle g^{ik}$.

3. This problem is in several parts each of which is helpful in solving the next.

(a) Let $k > 0$. Show that if a smooth function $f(x, y)$ satisfies $f(x, y) \ge k(x^2 + y^2)$ in a neighborhood of $(0, 0)$ and also we have $f(0, 0) = 0$, then each eigenvalue of the Hessian of f at $(0, 0)$ is at least $2k$.
(b) Find a lower bound for the Gaussian curvature of the graph of f at the point $(0, 0, f(0, 0))$.
(c) Let M be a smooth closed surface in $\mathbb{R}^3$. Assume that the point P of M furthest from the origin in $\mathbb{R}^3$ lies on the z-axis. Determine the sign of the Gaussian curvature of M at P.

(d) Determine the sign of the mean curvature of M at P.