

5 february 2007

Differential geometry 88-526  
Homework 6

1. This problem deals with curves in Euclidean space.

- (a) Define a regular curve in  $\mathbb{R}^3$ .
- (b) Define the arclength parameter.
- (c) Consider surfaces  $M_1, M_2 \subset \mathbb{R}^3$  defined by

$$M_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 169\},$$

and

$$M_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x = 5\}.$$

Consider the intersection  $C = M_1 \cap M_2$ . Find an arclength parametrisation of  $C$ .

- (d) Calculate the curvature of  $C$ .

2. This problem deals with surfaces in Euclidean space.

- (a) Define a regular surface.
- (b) Consider the surface  $M_3 \subset \mathbb{R}^3$  defined by

$$M_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 4\}.$$

Calculate the Weingarten map of  $M_3$ .

- (c) Calculate the Gaussian curvature function  $K(u^1, u^2)$  of  $M_3$ .

3. Consider the surface  $M_4 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 9\}$ .

- (a) Find a parametrisation of  $M_4$ .
- (b) Consider a curve  $\beta(s)$  on  $M_4$  such that the vector  $\beta''(s) \in \mathbb{R}^3$  is proportional to  $\beta(s)$  for every value of parameter  $s$ . In other words, the pair of vectors  $(\beta(s), \beta''(s))$  is linearly dependent. Find a differential equation satisfied by the curve.

4. In coordinates  $(u^1, u^2) = (x, y)$ , consider the metric  $\lambda(y)(dx^2 + dy^2)$ , where  $\lambda(y) = y^{-2}$ .

- (a) Calculate the symbol  $\Gamma_{11}^1$  of the metric.
- (b) Calculate the Gaussian curvature function  $K(x, y)$  of the metric.

5. Consider a lattice  $L \subset \mathbb{R}^2$  defined by  $L = \mathbb{Z} \oplus 2\mathbb{Z}$ , in other words,  $L = \{(n, 2m) \in \mathbb{R}^2 \mid n, m \in \mathbb{Z}\}$ .

- (a) Calculate the successive minima  $\lambda_i$  of  $L$ .
- (b) Calculate the systole  $\text{sys}\pi_1(\mathbb{T}_{1,2})$  of the torus  $\mathbb{T}_{1,2} = \mathbb{R}^2/L$ .
- (c) Define conformal equivalence of metrics.

- (d) Find the smallest constant  $C > 0$  such that for every torus  $\mathbb{T}$  conformally equivalent to  $\mathbb{T}_{1,2}$ , one has  $\text{sys}\pi_1(\mathbb{T})^2 \leq C \text{area}(\mathbb{T})$ .