1. Differential geometry 88-826-01 homework set 0

This is related to the material in chapter 13, section 13.1 on derivations, starting on page 105, on the notion of a derivation.

Proposition 13.1.5 on page 106 proves that the dimension of the space of derivations on euclidean n-space is precisely n. The proof provided on page 106 deals only with the 1-dimensional case.

1. Give an explicit proof of Proposition 13.1.5 in the 2-dimensional case.

2. Differential geometry 88-826-01 homework set 1

1. Consider the parabola

$$x = z^2 + \frac{1}{4}$$

in the xz-plane. Let M be the surface of revolution obtained by rotating the parabola around the z-axis.

(a) find a parametrisation of the parabola;
(b) find a parametrisation of M;
(c) find the ratio $\frac{\kappa_1}{\kappa_2}$ of the principal curvatures of M.

2. Let $x(u^1, u^2)$ be a parametrized surface in $\mathbb{R}^3$. Find an expression for the scalar product $\langle x_{ij}, x_{kl} \rangle$ in terms of a combination of the following data: the Γ_{ij}^k symbols, the coefficients of the first fundamental form, and the coefficients of the second fundamental form.

3. Let C be the curve in the (x, z)-plane which is the locus of the equation $(x - 3)^2 + z^2 = 1$.

(a) Find a unit speed parametrisation of C;
(b) find a parametrisation $x(\theta, \phi)$ of the surface of revolution $M \subset \mathbb{R}^3$ obtained by rotating C around the z-axis;
(c) calculate the coefficients of the first and second fundamental forms;
(d) calculate the Gaussian curvature $K(\theta, \phi)$;
(e) determine when the Gaussian curvature is positive;
(f) describe geometrically the region on M where the Gaussian curvature is positive.
(g) describe geometrically the boundary between the region on M where the Gaussian curvature is positive, and the region where it is negative.

4. This problem concerns the calculation of Gaussian curvature K.

(a) Describe at least four possible ways of calculating K.
(b) Which of the approaches in (a) are applicable if the data one is given is that the metric is defined in coordinates (u^1, u^2) by the metric coefficients $g_{ij}(u^1, u^2) = \frac{1}{(u^2)^2}\delta_{ij}$ but one is *not* given any explicit imbedding in Euclidean space?
(c) Calculate K for the metric in (b).

3. Differential geometry 88-826 homework set 2

1. Given a metric g on a torus, let $\tau \in D$ be its conformal parameter, where $D \subset \mathbb{C}$ is the standard fundamental domain

$$D = \{z = x + iy \in \mathbb{C} : |x| \leq \tfrac{1}{2}, y > 0, |z| \geq 1\}.$$

Thus, g is conformally equivalent to a flat metric $\mathbb{C}/L$ where the lattice L is spanned by $\tau \in \mathbb{C}$ and $1 \in \mathbb{C}$, so that $L = \mathbb{Z}\tau + \mathbb{Z}1$. If g is a torus of revolution in $\mathbb{R}^3$, let λ_φ be the length of the φ-loop, let $\lambda_{\theta_{\min}}$ be the least length of a θ-loop, and let $\lambda_{\theta_{\max}}$ be the maximal length of a θ-loop (see Section ?? of the course notes). Consider the torus of revolution whose generating curve is the circle in the (x, z) plane centered at the point with coordinates $(3, 4)$ and of radius 2.

(a) Find λ_φ of the torus.
(b) Find $\lambda_{\theta_{\min}}$ of the torus.
(c) Find $\lambda_{\theta_{\max}}$ of the torus.
(d) Find the conformal parameter τ of the torus.

2. Let $x(u^1, u^2)$ be a parametrized surface in 3-space. Express the following quantities in terms of the coefficents g_{ij} of the first fundamental form; the symbols Γ^k_{ij}; the coefficents $L^i{}_j$ of the Weingarten map; and the coefficients L_{ij} of the second fundamental form, simplifying the expression as much as possible. Here the Einstein summation convention implies summation over every index occurring both in a lower position and in an upper position.

(a) $\langle x_{\ell j}, x_k \rangle \left(\delta^k{}_m\right) g^{m\ell}$.
(b) $\langle n_j, x_{pq} \rangle \left(\delta^j{}_r\right)$.
(c) $\langle x_{stu}, n \rangle$.
(d) $g_{pq} \left(\delta^q{}_s\right) g^{su} \delta^p{}_u$.

3. This problem and problem 4 are in several parts each of which is helpful in solving the next.

(a) Let $k > 0$. Show that if a smooth function $f(x, y)$ satisfies $f(x, y) \geq k(x^2 + y^2)$ in a neighborhood of $(0, 0)$ and also we have $f(0, 0) = 0$, then each eigenvalue of the Hessian of f at $(0, 0)$ is at least $2k$.
(b) Find a lower bound for the Gaussian curvature of the graph of f at the point $(0, 0, f(0, 0))$.

4. This problem is a continuation of problem 3.

(a) Let M be a smooth closed surface in $\mathbb{R}^3$. Assume that the point P of M furthest from the origin in $\mathbb{R}^3$ lies on the z-axis. Prove that the normal vector at the point P is proportional to its position vector (radius-vector).
(b) Determine the sign of the Gaussian curvature of M at P and give a lower bound for its absolute value.
(c) Give a bound for the mean curvature of M at P.

4. Differential geometry 88-826 homework set 3

1. Consider a regular surface $\underline{x}(u^1, u^2)$ in $\mathbb{R}^3$.

(a) Define what is meant by the regularity of $\underline{x}(u^1, u^2)$.

(b) Prove that the expression $\frac{\partial}{\partial u^m}\left(\Gamma_{ij}^k x_k + L_{ij} n\right)$ is symmetric with respect to j and m.

(c) Write the expression $L_{i[j} L_{k]}^q$ in terms of the Γ symbols alone.

(d) Let $\beta = x \circ \alpha$ be a curve on the surface. Assume that for all $t > 0$, the vector $\beta''(t)$ is proportional to $x_1 \times x_2$. Find an ordinary differential equation satisfied by components $\alpha^k(t)$ of α.

2. Let $\rho > 0$ be a real number. Consider the metric

$$\frac{1}{\left(1 + \frac{\rho}{4}(x^2 + y^2)\right)^2}(dx^2 + dy^2),$$

namely the metric obtained from the standard flat metric by multiplying by the conformal factor $\lambda = f^2$ where

$$f(x, y) = \left(1 + \frac{\rho}{4}(x^2 + y^2)\right)^{-1}.$$

(a) Specify which methods of calculating Gaussian curvature are applicable.

(b) Calculate the Gaussian curvature of the metric.

3. Let $z = \sqrt{9 - x^2}$ be a curve in the (x, z) plane.

(a) Find a parametrisation of the corresponding surface of revolution M in $\mathbb{R}^3$.

(b) calculate the mean curvature of M.

4. Consider a torus $T^2 = \mathbb{C}/L$.

(a) Define the parameter $\tau(T^2)$.

(b) Describe geometrically what it means for $\tau(T^2)$ to be pure imaginary and specify a fundamental domain for T^2.

(c) Formulate Fubini's theorem for functions in the plane.

(d) Exploit Fubini's theorem to prove Loewner's inequality

$$\text{sys}^2 \leq \text{area}$$

for a torus with pure imaginary parameter τ.