Step 6 The curve is W-shaped, as shown in Figure 3.7.11.

10, 11

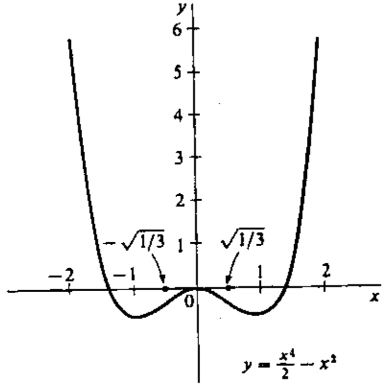

Figure 3.7.11

PROBLEMS FOR SECTION 3.7

Sketch each of the curves given below by the six-step process explained in the text. For each curve, give a table showing all the critical points, local maxima and minima, intervals on which the curve is increasing or decreasing, points of inflection, and intervals on which the curve is concave upward or downward.

1 $y = x^2 + 2, \quad -2 \le x \le 2$

2 $y = 1 - x^2, \quad -2 \le x \le 2$

3 $y = x^2 - 2x, \quad -2 \le x \le 2$

4 $y = \frac{1}{2}x^2 + x, \quad -2 \le x \le 2$

5 $y = 2x^2 - 4x + 3, \quad 0 \le x \le 2$

6 $y = -x^2 - 2x + 6, \quad -4 \le x \le 0$

7 $y = x^4, \quad -2 \le x \le 2$

8 $y = x^5, \quad -2 \le x \le 2$

9 $y = x^3 + x^2 + x, \quad -2 \le x \le 2$

10 $y = x^3 + x^2 - x, \quad -2 \le x \le 2$

11 $y = \frac{1}{3}x^3 + x^2 + x, \quad -2 \le x \le 2$

12 $y = -x^3 + 12x - 12, \quad -3 \le x \le 3$

13 $y = x^4 + 4x^3 + 2, \quad -4 \le x \le 2$

14 $y = \frac{1}{4}x^4 - x, \quad -2 \le x \le 2$

15 $y = x^2 - \frac{1}{2}x^4, \quad -2 \le x \le 2$

16 $y = x^2(x-2)^2, \quad -1 \le x \le 3$

17 $y = 1/x, \quad -4 \le x \le -\frac{1}{4}$ and $\frac{1}{4} \le x \le 4$

18 $y = 1/x + x, \quad -4 \le x \le -\frac{1}{4}$ and $\frac{1}{4} \le x \le 4$

19 $y = x^{-2}, \quad -2 \le x \le -\frac{1}{2}$ and $\frac{1}{2} \le x \le 2$

20 $y = x + x^{-2}, \quad -2 \le x \le -\frac{1}{2}$ and $\frac{1}{2} \le x \le 2$

21 $y = \dfrac{x-1}{x+1}, \quad 0 \le x \le 10$

22 $y = \dfrac{2x}{x+1}, \quad 0 \le x \le 10$

23 $y = \dfrac{1}{x^2+1}, \quad -4 \le x \le 4$

24 $y = \dfrac{x}{x^2+1}, \quad -4 \le x \le 4$

25 $y = \dfrac{x^2}{x^2+1}, \quad -2 \le x \le 2$

26 $y = \dfrac{1}{x^2-1}, \quad -\frac{9}{10} \le x \le \frac{9}{10}$

27 $y = \sqrt{x}, \quad \frac{1}{4} \le x \le 4$

28 $y = 2\sqrt{x} - x, \quad \frac{1}{4} \le x \le 4$

29 $y = 1/\sqrt{x}, \quad \frac{1}{4} \le x \le 4$

30 $y = x^{1/2} + x^{-1/2}, \quad \frac{1}{4} \le x \le 4$

31 $y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$

32 $y = \sqrt{9 + x^2}, \quad -4 \le x \le 4$

33 $y = \sin x \cos x, \quad 0 \le x \le 2\pi$

34 $y = \sin x + \cos x, \quad 0 \le x \le 2\pi$

35 $y = 3\sin(\frac{1}{2}x), \quad 0 \le x \le 2\pi$

36 $y = \sin^2 x, \quad 0 \le x \le 2\pi$

37 $y = \tan x, \quad -\pi/3 \le x \le \pi/3$

38 $y = 1/\cos x, \quad -\pi/3 \le x \le \pi/3$

39 $y = e^{-x}, \quad -2 \le x \le 2$

40 $y = e^{(1/2)x}, \quad -2 \le x \le 2$

41 $y = \ln x, \quad 1/e \le x \le e$

42 $y = (\ln x)^2, \quad 1/e \le x \le e$

43 $y = xe^{-x}, \quad -1 \le x \le 3$

44 $y = x - e^x, \quad -2 \le x \le 2$

45 $y = x \ln x, \quad e^{-2} \le x \le e$

46 $y = x - \ln x, \quad e^{-2} \le x \le e$

47 $y = xe^x, \quad -3 \le x \le 1$

48 $y = e^{-x^2}, \quad -2 \le x \le 2$

49 $y = e^x/x, \quad \frac{1}{4} \le x \le 4$

50 $y = \ln(1 + x^2), \quad -3 \le x \le 3$

3.8 PROPERTIES OF CONTINUOUS FUNCTIONS

This section develops some theory that will be needed for integration in Chapter 4. We begin with a new concept, that of a *hyperinteger*. The hyperintegers are to the integers as the hyperreal numbers are to the real numbers. The hyperintegers consist of the ordinary finite integers, the positive infinite hyperintegers, and the negative infinite hyperintegers. The hyperintegers have the same algebraic properties as the integers and are spaced one apart all along the hyperreal line as in Figure 3.8.1.

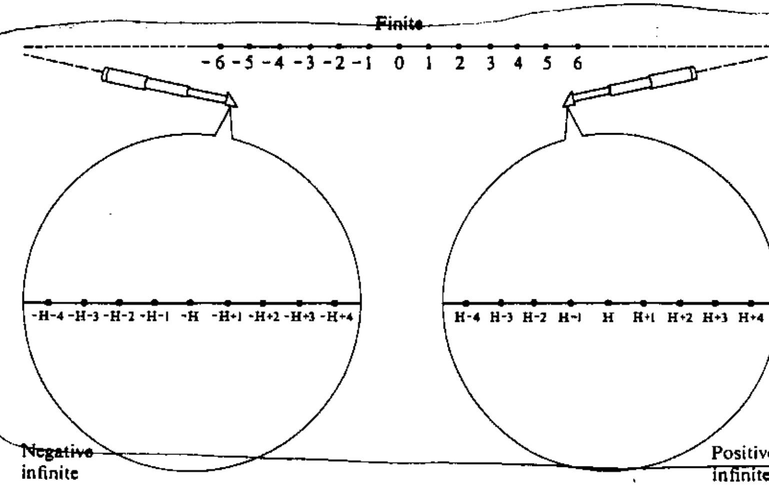

Figure 3.8.1 The Set of Hyperintegers

The rigorous definition of the hyperintegers uses the greatest integer function $[x]$ introduced in Section 3.4, Example 6. Remember that for a real number x, $[x]$ is the greatest integer n such that $n \le x$. A real number y is itself an integer if and only if $y = [x]$ for some real x. To get the hyperintegers, we apply the function $[x]$ to hyperreal numbers x (see Figure 3.8.2).

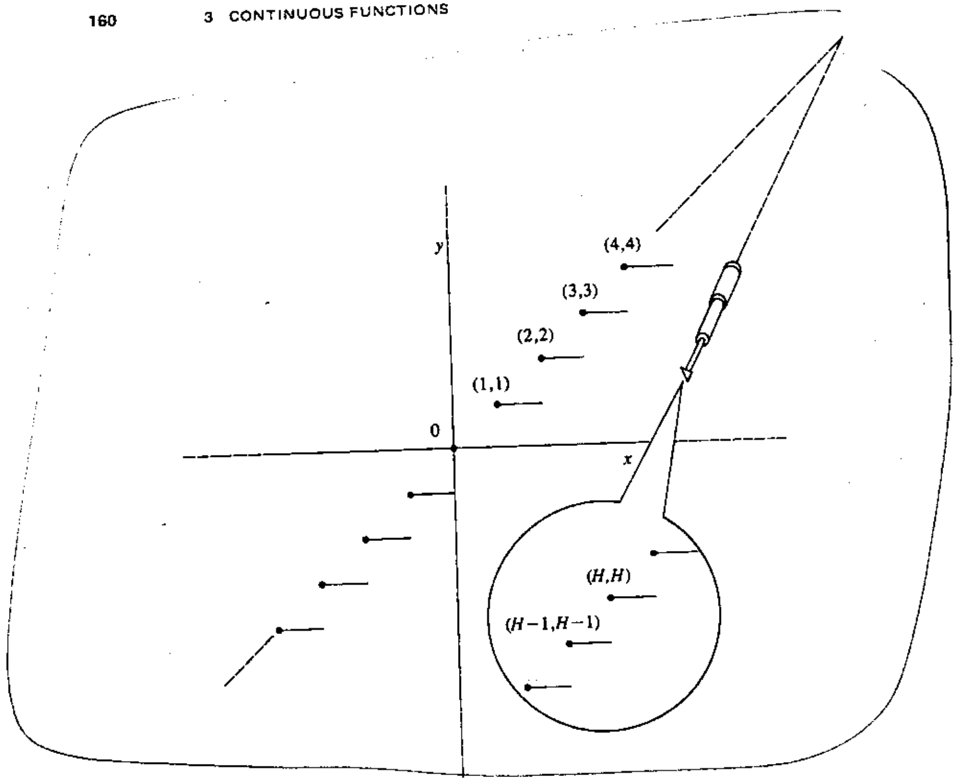

Figure 3.8.2

DEFINITION

*A **hyperinteger** is a hyperreal number y such that $y = [x]$ for some hyperreal x.*

When x varies over the hyperreal numbers, $[x]$ is the greatest hyperinteger y such that $y \le x$. Because of the Transfer Principle, every hyperreal number x is between two hyperintegers $[x]$ and $[x] + 1$,

$$[x] \le x < [x] + 1.$$

Also, sums, differences, and products of hyperintegers are again hyperintegers.

We are now going to use the hyperintegers. In sketching curves we divided a closed interval $[a, b]$ into finitely many subintervals. For theoretical purposes in the calculus we often divide a closed interval into a finite or infinite number of equal subintervals. This is done as follows.

Given a closed real interval $[a, b]$, a *finite partition* is formed by choosing a positive integer n and dividing $[a, b]$ into n equal parts, as in Figure 3.8.3. Each part will be a subinterval of length $t = (b - a)/n$. The n subintervals are

$$[a, a + t], [a + t, a + 2t], \ldots, [a + (n - 1)t, b].$$

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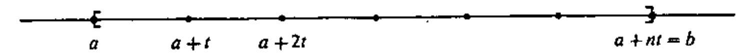


Figure 3.8.3

The endpoints

$$a, a + t, a + 2t, \ldots, a + (n - 1)t, a + nt = b$$

are called *partition points.*

The real interval $[a, b]$ is contained in the *hyperreal interval* $[a, b]^*$, which is the set of all hyperreal numbers x such that $a \le x \le b$. An infinite partition is applied to the hyperreal interval $[a, b]^*$ rather than the real interval. To form an infinite partition of $[a, b]^*$, choose a positive infinite hyperinteger H and divide $[a, b]^*$ into H equal parts as shown in Figure 3.8.4. Each subinterval will have the same infinitesimal length $\delta = (b - a)/H$. The H subintervals are

$$[a, a + \delta], [a + \delta, a + 2\delta], \ldots, [a + (K - 1)\delta, a + K\delta], \ldots, [a + (H - 1)\delta, b],$$

and the partition points are

$$a, a + \delta, a + 2\delta, \ldots, a + K\delta, \ldots, a + H\delta = b,$$

where K runs over the hyperintegers from 1 to H. Every hyperreal number x between a and b belongs to one of the infinitesimal subintervals,

$$a + (K - 1)\delta \le x < a + K\delta.$$

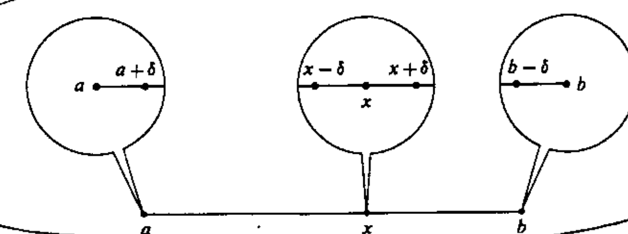


Figure 3.8.4 An infinite partition

We shall now use infinite partitions to sketch the proofs of three basic results, called the Intermediate Value Theorem, the Extreme Value Theorem, and Rolle's Theorem. The use of these results will be illustrated by studying zeros of continuous functions. By a *zero* of a function f we mean a point c where $f(c) = 0$. As we can see in Figure 3.8.5, the zeros of f are the points where the curve $y = f(x)$ intersects the x-axis.

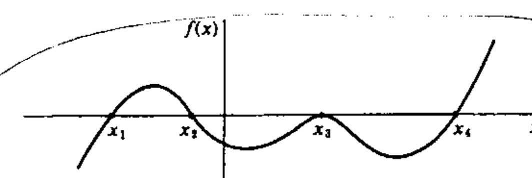


Figure 3.8.5

ערך בינ"ם

INTERMEDIATE VALUE THEOREM

Suppose the real function f is continuous on the closed interval $[a, b]$ and $f(x)$ is positive at one endpoint and negative at the other endpoint. Then f has a zero in the interval (a, b); that is, $f(c) = 0$ for some real c in (a, b).

Discussion There are two cases illustrated in Figure 3.8.6:

$$f(a) < 0 < f(b) \quad \text{and} \quad f(a) > 0 > f(b).$$

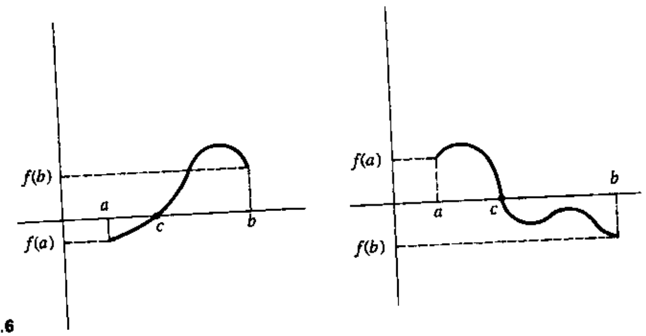


Figure 3.8.6

In the first case, the theorem says that if a continuous curve is below the x-axis at a and above it at b, then the curve must intersect the x-axis at some point c between a and b. Theorem 3 in the preceding Section 3.7 on curve sketching is simply a reformulation of the Intermediate Value Theorem.

SKETCH OF PROOF We assume $f(a) < 0 < f(b)$. Let H be a positive infinite hyperinteger and partition the interval $[a, b]^*$ into H equal parts

$$a, a + \delta, a + 2\delta, \ldots, a + H\delta = b.$$

Let $a + K\delta$ be the last partition point at which $f(a + K\delta) < 0$. Thus

$$f(a + K\delta) < 0 \le f(a + (K + 1)\delta).$$

Since f is continuous, $f(a + K\delta)$ is infinitely close to $f(a + (K + 1)\delta)$. We conclude that $f(a + K\delta) \approx 0$ (Figure 3.8.7). We take c to be the standard part of $a + K\delta$, so that

$$f(c) = st(f(a + K\delta)) = 0.$$

EXAMPLE 1 The function

$$f(x) = \frac{1}{1 + x} - x - \sqrt{x} - \sqrt[3]{x},$$

which is shown in Figure 3.8.8, is continuous for $0 \le x \le 1$. Moreover,

$$f(0) = 1, \qquad f(1) = \tfrac{1}{2} - 3 = -2\tfrac{1}{2}.$$

The Intermediate Value Theorem shows that $f(x)$ has a zero $f(c) = 0$ for some c between 0 and 1.

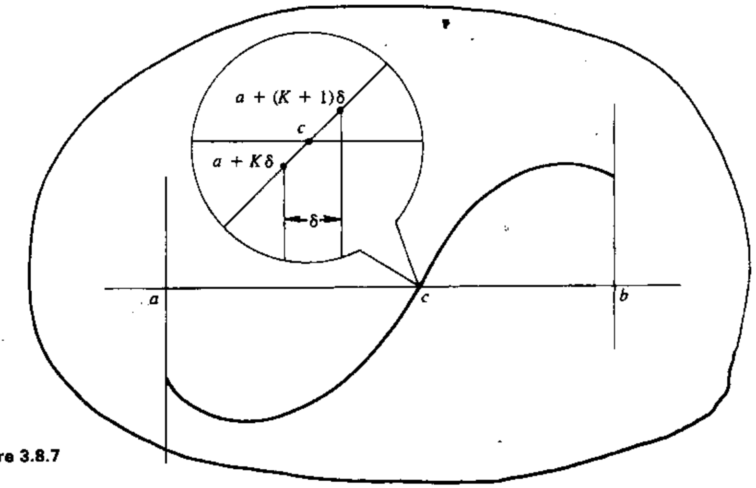


Figure 3.8.7

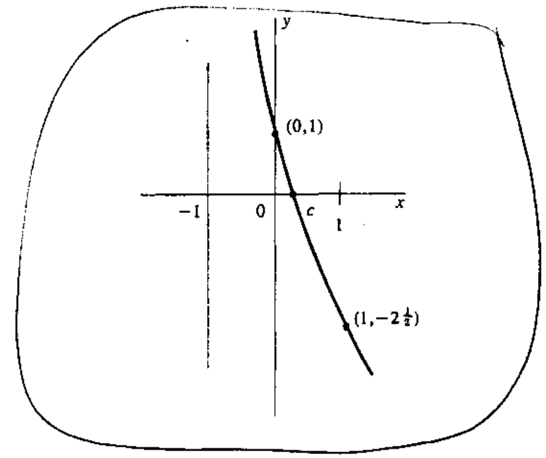


Figure 3.8.8

The Intermediate Value Theorem can be used to prove Theorem 3 of Section 3.7 on curve sketching:

Suppose g is a continuous function on an interval I, and $g(x) \ne 0$ for all x in I.

(i) *If $g(c) > 0$ for at least one c in I, then $g(x) > 0$ for all x in I.*
(ii) *If $g(c) < 0$ for at least one c in I, then $g(x) < 0$ for all x in I.*

PROOF (i) Let $g(c) > 0$ for some c in I. If $g(x_1) < 0$ for some other point x_1 in I, then by the Intermediate Value Theorem there is a point x_2 between c and x_1 such that $g(x_2) = 0$, contrary to hypothesis (Figure 3.8.9). Therefore we conclude that $g(x) > 0$ for all x in I.

Recall (1) Critical point thm ($f'(c)=0$ at max)

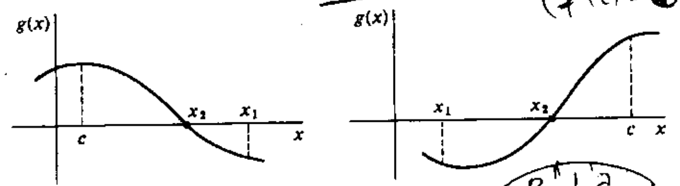


Figure 3.8.9

(2) Intermediate value theorem ($f(c)=0$).

(3)

EXTREME VALUE THEOREM

Let f be continuous on its domain, which is a closed interval $[a, b]$. Then f has a maximum at some point in $[a, b]$, and a minimum at some point in $[a, b]$.

Discussion We have seen several examples of functions that do not have maxima on an *open* interval, such as $f(x) = 1/x$ on $(0, \infty)$, or $g(x) = 2x$ on $(0, 1)$. The Extreme Value Theorem says that on a closed interval a continuous function always has a maximum.

SKETCH OF PROOF Form an infinite partition of $[a, b]^*$,

$$a, a + \delta, a + 2\delta, \ldots, a + H\delta = b.$$

do you use boundedness here?

By the Transfer Principle, there is a partition point $a + K\delta$ at which $f(a + K\delta)$ has the largest value. Let c be the standard part of $a + K\delta$ (see Figure 3.8.10). Any point u of $[a, b]^*$ lies in a subinterval, say

$$a + L\delta \le u < a + (L + 1)\delta.$$

We have

$$f(a + K\delta) \ge f(a + L\delta),$$

and taking standard parts,

$$f(c) \ge f(u).$$

This shows that f has a maximum at c.

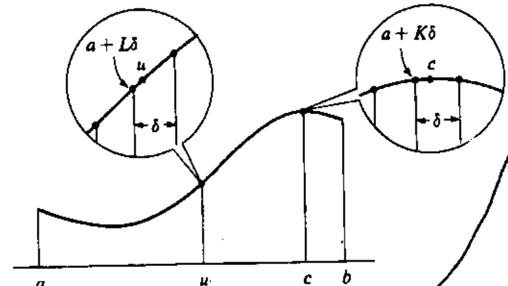


Figure 3.8.10 Proof of the Extreme Value Theorem

NB. Write this also as $\inf_{a \le x \le b} f(x) = \min_{a \le x \le b} f(x)$, $\sup_{a \le x \le b} f(x) = \max$ etc

Corollary: If f cont then $f([a,b])$ is a closed interval. Proof: extreme + intermediate.

29 may

ROLLE'S THEOREM

Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If

$$f(a) = f(b) = 0,$$

then there is at least one point c strictly between a and b where f has derivative zero; i.e.,

$$f'(c) = 0 \quad \text{for some } c \text{ in } (a, b).$$

Geometrically, the theorem says that a differentiable curve touching the x-axis at a and b must be horizontal for at least one point strictly between a and b.

PROOF We may assume that $[a, b]$ is the domain of f. By the Extreme Value Theorem, f has a maximum value M and a minimum value m in $[a, b]$. Since $f(a) = 0$, $m \le 0$ and $M \ge 0$ (see Figure 3.8.11).

Case 1 $M = 0$ and $m = 0$. Then f is the constant function $f(x) = 0$, and therefore $f'(c) = 0$ for all points c in (a, b).

Case 2 $M > 0$. Let f have a maximum at c, $f(c) = M$. By the Critical Point Theorem, f has a critical point at c. c cannot be an endpoint because the value of $f(x)$ is zero at the endpoints and positive at $x = c$. By hypothesis, $f'(x)$ exists at $x = c$. It follows that c must be a critical point of the type $f'(c) = 0$.

Case 3 $m < 0$. We let f have a minimum at c. Then as in Case 2, c is in (a, b) and $f'(c) = 0$.

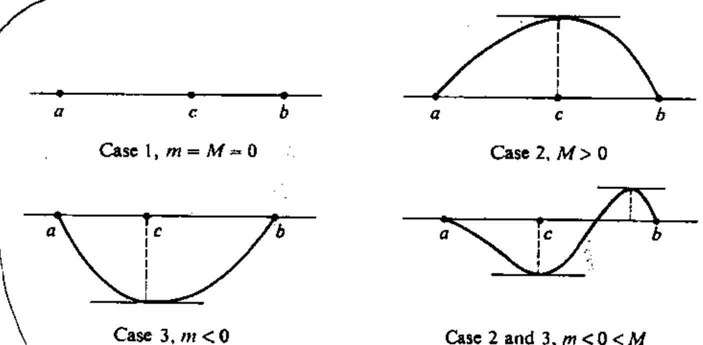


Figure 3.8.11 Rolle's Theorem

EXAMPLE 2 $f(x) = (x - 1)^2(x - 2)^3$, $a = 1$, $b = 2$. The function f is continuous and ~~differentiable everywhere~~ (Figure 3.8.12). Moreover, $f(1) = f(2) = 0$. Therefore by Rolle's Theorem there is a point c in $(1, 2)$ with $f'(c) = 0$.

Let us find such a point c. We have

$$f'(x) = 3(x - 1)^2(x - 2)^2 + 2(x - 1)(x - 2)^3 = (x - 1)(x - 2)^2(5x - 7).$$

$f(c)$ when $c \in [a,b]$.

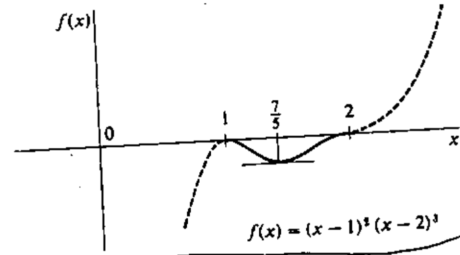


Figure 3.8.12

Notice that $f'(1) = 0$ and $f'(2) = 0$. But Rolle's Theorem says that there is another point c which is in the *open* interval $(1, 2)$ where $f'(c) = 0$. The required value for c is $c = \frac{7}{5}$ because $f'(\frac{7}{5}) = 0$ and $1 < \frac{7}{5} < 2$.

EXAMPLE 3 Let $f(x) = \dfrac{x^4}{2} - x^2,\quad a = -\sqrt{2},\quad b = \sqrt{2}.$

Then $f(a) = f(b) = 0$.
Rolle's Theorem says that there is at least one point c in $(-\sqrt{2}, \sqrt{2})$ at which $f'(c) = 0$. As a matter of fact there are three such points,

$$c = -1, \quad c = 0, \quad c = 1.$$

We can find these points as follows:

$$f'(x) = 2x^3 - 2x = 2x(x^2 - 1),$$
$$f'(x) = 0 \quad \text{when} \quad x = 0 \quad \text{or} \quad x = \pm 1.$$

The function is drawn in Figure 3.8.13.

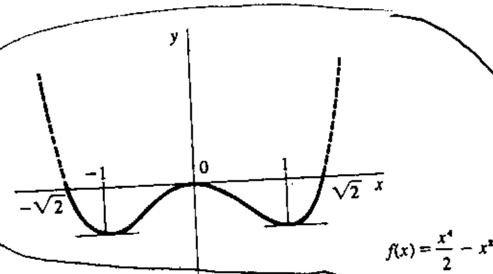


Figure 3.8.13

EXAMPLE 4 $f(x) = \sqrt{1 - x^2}$, $a = -1$, $b = 1$. Then $f(-1) = f(1) = 0$. The func- ~~tion f is continuous on $[-1, 1]$ and has a~~ derivative at each point of $(-1, 1)$, as Rolle's Theorem requires (Figure 3.8.14). Note, however, that $f'(x)$ does not exist at either endpoint, $x = -1$ or $x = 1$. By Rolle's Theorem there is a point c in $(-1, 1)$ such that $f'(c) = 0$, $c = 0$ is such a point, because

$$f'(x) = -\frac{x}{\sqrt{1 - x^2}}, \qquad f'(0) = 0.$$

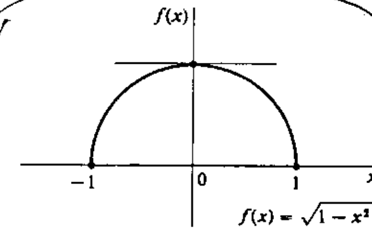


Figure 3.8.14

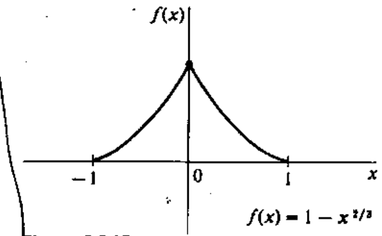


Figure 3.8.15

EXAMPLE 5 $f(x) = 1 - x^{2/3}$, $a = -1$, $b = 1$. Then $f(-1) = f(1) = 0$, and $f'(x) = -\frac{2}{3}x^{-1/3}$ for $x \neq 0$. $f'(0)$ is undefined. There is no point c in $(-1, 1)$ at which $f'(c) = 0$. Rolle's Theorem does not apply in this case because $f'(x)$ does not exist at one of the points of the interval $(-1, 1)$, namely at $x = 0$. In Figure 3.8.15, we see that instead of being horizontal at a point in the interval, the curve has a sharp peak.

Rolle's Theorem is useful in finding the number of zeros of a differentiable function f. It shows that between any two zeros of f there must be one or more zeros of f'. It follows that if f' has no zeros in an interval I, then f cannot have more than one zero in I.

EXAMPLE 6 How many zeros does the function $f(x) = x^3 + x + 1$ have? We use both Rolle's Theorem and the Intermediate Value Theorem.

Using Rolle's Theorem: $f'(x) = 3x^2 + 1$. For all x, $x^2 \geq 0$, and hence $f'(x) \geq 1$. Therefore $f(x)$ has at most one zero.

Using the Intermediate Value Theorem: We have $f(-1) = -1$, $f(0) = 1$. Therefore f has at least one zero between -1 and 0.

CONCLUSION f has exactly one zero, and it lies between -1 and 0 (see Figure 3.8.16).

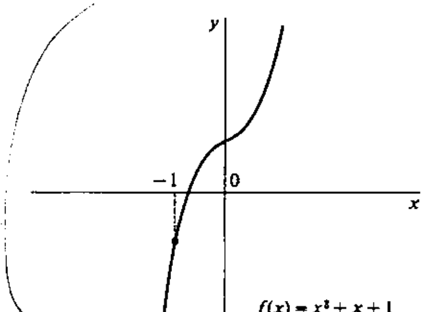


Figure 3.8.16

Our method of sketching curves in Section 3.7 depends on a consequence of Rolle's Theorem called the Mean Value Theorem. It deals with the average slope of a curve between two points.

DEFINITION

Let f be defined on the closed interval $[a, b]$. The ***average slope*** *of f between a and b is the quotient*

$$\text{average slope} = \frac{f(b) - f(a)}{b - a}.$$

We can see in Figure 3.8.17 that the average slope of f between a and b is equal to the slope of the line passing through the points $(a, f(a))$ and $(b, f(b))$. This is shown by the two-point equation for a line (Section 1.3). In particular, if f is already a linear function $f(x) = mx + c$, then the average slope of f between a and b is equal to the slope m of the line $y = f(x)$.

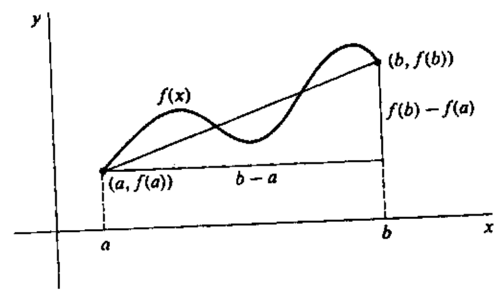


Figure 3.8.17 Average Slope

This is shown by the two-point equation for a straight line (Section 1.2). In particular, if f is already a linear function $f(x) = mx + c$, then the average slope of f between a and b is equal to the slope m of the straight line $y = f(x)$.

Lagrange's theorem
mishpat ha'erech hamemutzah

MEAN VALUE THEOREM

Assume that f is continuous on the closed interval $[a, b]$ and has a derivative at every point of the open interval (a, b). Then there is at least one point c in (a, b) where the slope $f'(c)$ is equal to the average slope of f between a and b,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remark In the special case that $f(a) = f(b) = 0$, the Mean Value Theorem becomes Rolle's Theorem:

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{b - a} = 0.$$

On the other hand, we shall use Rolle's Theorem in the proof of the Mean Value Theorem. The Mean Value Theorem is illustrated in Figure 3.8.18.

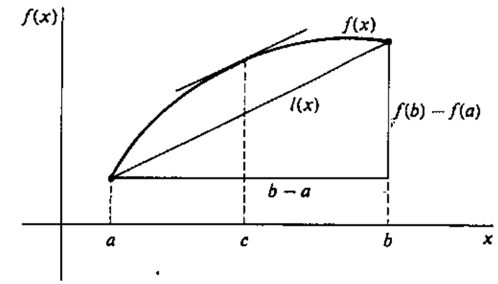


Figure 3.8.18 The Mean Value Theorem

PROOF OF THE MEAN VALUE THEOREM Let m be the average slope, $m = (f(b) - f(a))/(b - a)$. The line through the points $(a, f(a))$ and $(b, f(b))$ has the equation

$$l(x) = f(a) + m(x - a).$$

Let $h(x)$ be the vertical distance of $f(x)$ above $l(x)$,

$$h(x) = f(x) - l(x).$$

Then h is continuous on $[a, b]$ and has the derivative

$$h'(x) = f'(x) - l'(x) = f'(x) - m$$

at each point in (a, b). Since $f(x) = l(x)$ at the endpoints a and b, we have

$$h(a) = 0, \qquad h(b) = 0.$$

Therefore Rolle's Theorem can be applied to the function h, and there is a point c in (a, b) such that $h'(c) = 0$. Thus

$$0 = h'(c) = f'(c) - l'(c) = f'(c) - m,$$

whence

$$f'(c) = m.$$

We can give a physical interpretation of the Mean Value Theorem in terms of velocity. Suppose a particle moves along the y-axis according to the equation $y = f(t)$. The *average velocity* of the particle between times a and b is the ratio

$$\frac{f(b) - f(a)}{b - a}$$

of the change in position to the time elapsed. The Mean Value Theorem states that there is a point of time c, $a < c < b$, when the velocity $f'(c)$ of the particle is equal to the average velocity between times a and b.

Theorems 1 and 2 in Section 3.7 on curve sketching are consequences of the Mean Value Theorem. As an illustration, we prove part (ii) of Theorem 1:

Thm *If $f'(x) > 0$ for all interior points x of I, then f is increasing on I.*

PROOF Let $x_1 < x_2$ where x_1 and x_2 are points in I. By the Mean Value Theorem there is a point c strictly between x_1 and x_2 such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since c is an interior point of I, $f'(c) > 0$. Because $x_1 < x_2$, $x_2 - x_1 > 0$. Thus

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0, \quad f(x_2) - f(x_1) > 0, \quad f(x_2) > f(x_1).$$

This shows that f is increasing on I.

NB. Function may be strictly monotone increasing but there is a point where $f'=0$. Even though all ratios are positive: x^3.

PROBLEMS FOR SECTION 3.8

In Problems 1–16, use the Intermediate Value Theorem to show that the function has at least one zero in the given interval.

1 $f(x) = x^4 - 2x^3 - x^2 + 1, \quad 0 \le x \le 1$

2 $f(x) = x^2 + x - 3/x, \quad 1 \le x \le 2$

3 $f(x) = \sqrt{x} + \sqrt{x+1} - x, \quad 4 \le x \le 9$

4 $f(x) = \sqrt{x} + 1/x^2 - x^2, \quad 1 \le x \le 2$

5 $f(x) = \dfrac{2}{1 + x\sqrt{x}} - \sqrt{x^2+2}, \quad 0 \le x \le 1$

6 $f(x) = x^5 + x - \sqrt{x+1}, \quad 0 \le x \le 1$

7 $f(x) = x^3 + x^2 - 1, \quad 0 \le x \le 1$

8 $f(x) = x^2 + 1 - \dfrac{3}{x+1}, \quad 0 \le x \le 1$

9 $f(x) = 1 - 3x + x^3, \quad 0 \le x \le 1$

10 $f(x) = 1 - 3x + x^3, \quad 1 \le x \le 2$

11 $f(x) = x^2 + \sqrt{x} - 1, \quad 0 \le x \le 1$

12 $f(x) = x^2 - (x+1)^{-1/2}, \quad 0 \le x \le 1$

13 $f(x) = \cos x - \frac{1}{10}, \quad 0 \le x \le \pi$

14 $f(x) = \sin x - 2\cos x, \quad 0 \le x \le \pi$

15 $f(x) = \ln x - \dfrac{1}{x}, \quad 1 \le x \le e$

16 $f(x) = e^x - 10x, \quad 1 \le x \le 10$

In Problems 17–30, determine whether or not f' has a zero in the interval (a, b). *Warning*: Rolle's Theorem may give a wrong answer unless all the hypotheses are met.

17 $f(x) = 5x^2 - 8x, \quad [a,b] = [0, \frac{8}{5}]$

18 $f(x) = 1 - x^{-2}, \quad [a,b] = [-1, 1]$

19 $f(x) = \sqrt{16 - x^4}, \quad [a,b] = [-2, 2]$

20 $f(x) = \sqrt{4 - x^{2/7}}, \quad [a,b] = [-128, 128]$

21 $f(x) = 1/x - x, \quad [a,b] = [-1, 1]$

22 $f(x) = (x-1)^2(x-2), \quad [a,b] = [1, 2]$

23 $f(x) = (x-4)^3x^4, \quad [a,b] = [0, 4]$

24 $f(x) = \dfrac{(x-2)(x-4)}{x^3 + x + 2}, \quad [a,b] = [2, 4]$

25 $f(x) = |x| - 1, \quad [a,b] = [-1, 1]$

26 $f(x) = \dfrac{x(x-2)}{x-1}, \quad [a,b] = [0, 2]$

27 $f(x) = x\sin x, \quad [a,b] = [0, \pi]$

28 $f(x) = e^x\cos x, \quad [a,b] = [-\pi/2, \pi/2]$

29 $f(x) = \tan x, \quad [a,b] = [0, \pi]$

30 $f(x) = \ln(1 - \sin x), \quad [a,b] = [0, \pi]$

31 Find the number of zeros of $x^4 + 3x + 1$ in $[-2, -1]$.

32 Find the number of zeros of $x^4 + 2x^3 - 2$ in $[0, 1]$.

33 Find the number of zeros of $x^4 - 8x - 4$.

34 Find the number of zeros of $2x + \sqrt{x} - 4$.

In Problems 35–42, find a point c in (a, b) such that $f(b) - f(a) = f'(c)(b - a)$.

35 $f(x) = x^2 + 2x - 1, \quad [a,b] = [0, 1]$

36 $f(x) = x^3, \quad [a,b] = [0, 3]$

37 $f(x) = x^{2/3}, \quad [a,b] = [0, 1]$

38 $f(x) = \sqrt{x+1}, \quad [a,b] = [0, 2]$

39 $f(x) = x + \sqrt{x}, \quad [a,b] = [0, 4]$

40 $f(x) = 2 + (1/x), \quad [a,b] = [1, 2]$

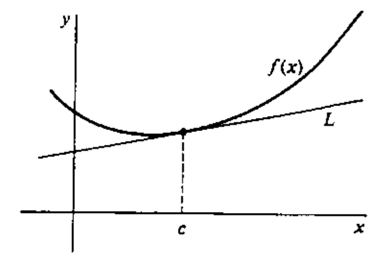
41 $f(x) = \dfrac{x-1}{x+1}, \quad [a,b] = [0, 2]$

42 $f(x) = x\sqrt{x+1}, \quad [a,b] = [0, 3]$

43 Use Rolle's Theorem to show that the function $f(x) = x^3 - 3x + b$ cannot have more than one zero in the interval $[-1, 1]$, regardless of the value of the constant b.

44 Suppose f, f', and f'' are all continuous on the interval $[a, b]$, and suppose f has at least three distinct zeros in $[a, b]$. Use Rolle's Theorem to show that f'' has at least one zero in $[a, b]$.

□ 45 Suppose that $f''(x) > 0$ for all real numbers x, so that the curve $y = f(x)$ is concave upward on the whole real line as illustrated in the figure. Let L be the tangent line to the curve at $x = c$. Prove that the line L lies below the curve at every point $x \ne c$.



EXTRA PROBLEMS FOR CHAPTER 3

1 Find the surface area A of a cube as a function of its volume V.

2 Find the length of the diagonal d of a rectangle as a function of its length x and width y.