www.math.wisc.edu/~keisler/calc.html

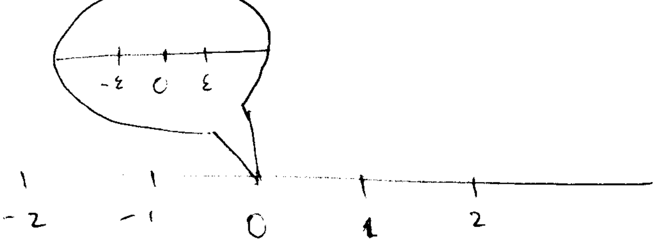
3''07

27 feb

(2) Calculus

$\mathbb{N} < \mathbb{Z} < \mathbb{Q} < \mathbb{R} < \mathbb{R}^*$ hyperreal

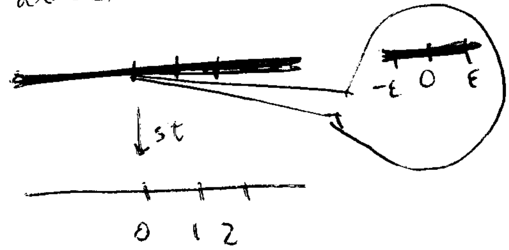
Recall order relation $<$ on $\mathbb{R}$: for any a, b, either $a < b$ or $a = b$ or $a > b$.



Keisler p.24 Def. A hyperreal number ε is called infinitesimal if

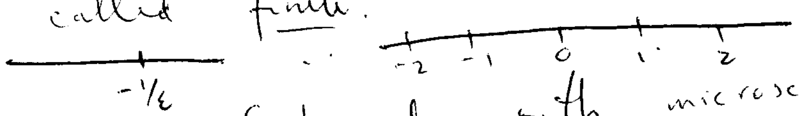
$$-a < \varepsilon < a$$

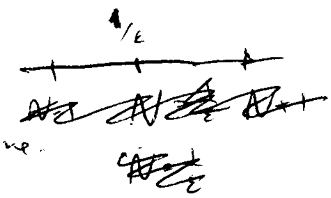
for every positive real number a.



If $\varepsilon > 0$ is infinitesimal, then $N = \frac{1}{\varepsilon}$ is positive infinite.

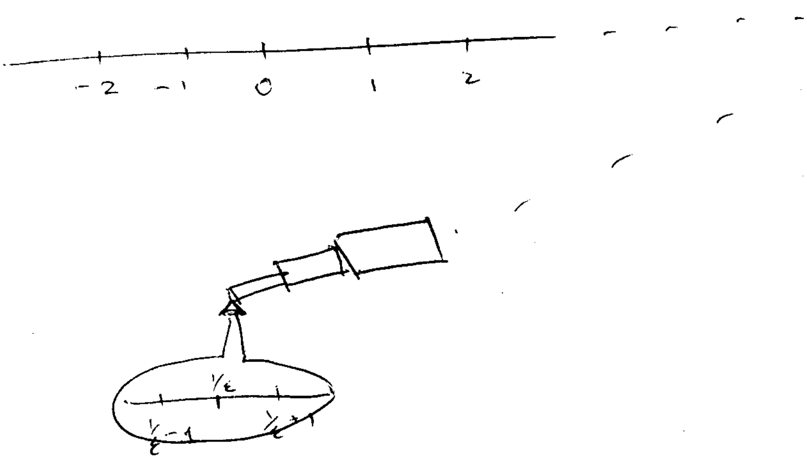
Hyperreal numbers which are not infinite numbers are called finite.

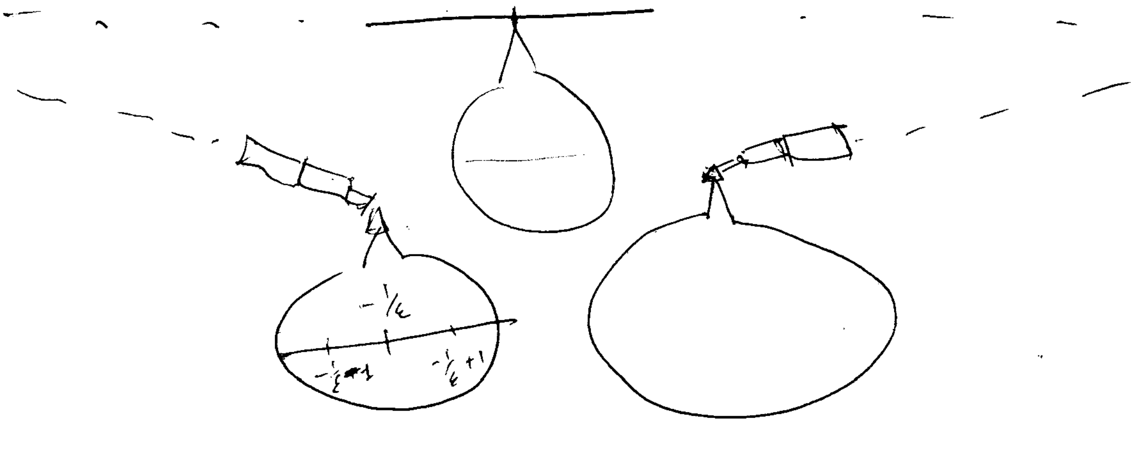




View infinitesimals with microscope: as above.

View infinite numbers with telescope





$\left\{ \begin{matrix} \text{finite} \\ \text{hyperreals} \end{matrix} \right\} \subset \mathbb{R}^*$

Example. calculation of slope of $y = x^2$ at x_0. $\frac{\Delta y}{\Delta x} = 2x + \Delta x$

conclusion: slope = real number infinitely close to $\frac{\Delta y}{\Delta x}$

Standard part ~~function~~:

$$st : \left\{ \begin{matrix} \text{finite} \\ \text{hyp} \end{matrix} \right\} \longrightarrow \mathbb{R}$$

$$2x + \Delta x \longmapsto 2x$$

$st(r)$ = real number infinitely close to r.

Extension principle
עקרון ההרחבה

(a) The real numbers ~~form~~ ~~part of~~ a subset of the hyperreal numbers, and the order relation $<$ for the real numbers is a subset of the order relation for the hyperreal numbers

(b) There is a hyperreal number that is greater than zero but less than every positive real number

(c) For every real function f of one variable or more variables, we are given a corresponding hyperreal function f^* of the same number of variables. f^* is called the natural extension of f.

27 feb

To explain (6),
let's give a careful definition of an infinitesimal.

<u>Def.</u> A hyperreal number b is said to be

<u>positive infinitesimal</u> if b is positive but less than every positive real number.

<u>negative infinitesimal</u> if b is negative but greater than every negative real number

<u>infinitesimal</u> if b is either positive infinitesimal, negative infinitesimal, or zero

now
① Recall $\mathbb{R} \subset \mathbb{R}^*$
② infinitesimal
③ extension princ.
④ microscope

<u>Examples of part c)</u> extension principle: since $+$ is a real function of two variables, its natural extension $+^*$ is a hyperreal function of two variables.
If x, y are hyperreal numbers, $x +^* y$ is formed by using the natural extension of $+$.
Similarly, product $x \cdot^* y$ is formed by using the natural extension of product function.

Part (c) extension principle also allows us to work with expression $\cos x$, $\sin(x + \cos(y))$. They mean $\cos^* x$, $\sin^*(x + \cos^*(y))$

The asterisks are <u>dropped</u>.

Transfer principle
עקרון הטרנספר

Every real statement (ממשית טענה) that holds for one or more particular real functions, holds for the hyperreal extension of these functions.

Examples of real statements

(1) closure law for addition: for any x and y, $x+y$ is defined

(2) Commutative law for addition: $x+y=y+x$

(3) a rule for order: if $0<x<y$ then $0<\frac{1}{y}<\frac{1}{x}$.
[this will be used on next page]

(4) Division by zero not allowed: $\frac{x}{0}$ undefined

(5) an algebraic identity: $(x-y)^2 = x^2-2xy+y^2$

(6) a trig identity: $\sin^2 x + \cos^2 x = 1$

(7) a rule for logarithms: if $x>0$ and $y>0$ then $\log_{10}(x\,y) = \log_{10} x + \log_{10} y$.

We can use the transfer principle to define the hyperreal functions:

(8) the square roots fcn is defined by the real statement: $y=\sqrt{x}$ if and only if $y^2=x$ and $y\geq$

27 feb

(9) The abs. value function is defined by the real statement $y = |x|$ iff $y = \sqrt{x^2}$.

[10] The common log is defined by the real statement

$y = \log_{10} x$ iff $10^y = x$.

Use these to construct lots of infinitesimals: $0 < \varepsilon^2 < \varepsilon$ (true for any $x < 1$)

in increasing order:

$\varepsilon^3, \quad \varepsilon^2, \quad \frac{\varepsilon}{100}, \quad \varepsilon, \quad 75\varepsilon, \quad \sqrt{\varepsilon}, \quad \varepsilon + \sqrt{\varepsilon}$.

---

17 oct '13

Definition A hyperreal number is said to be

(a) finite if b is between two real numbers.

(b) positive ~~finite~~ infinite if b is greater than every real number

(c) negative infinite if b is less than every real number.

Rules. If ε is infinitesimal, ~~then~~ and a is finite, then the product $\varepsilon \cdot a$ is ~~smaller~~ ~~if~~ infinitesimal.

Rule if ε positive infinitesimal, then $\frac{1}{\varepsilon}$ is positive infinite.

Let us prove it. Let r any positive real. Since ε is infinitesimal, $0 < \varepsilon < \frac{1}{r}$. Applying the transfer principle, $0 < r < \frac{1}{\varepsilon}$. Hence $\frac{1}{\varepsilon}$ is pos infinite

Rules for infinitesimal, finite, and infinite numbers

Assume that ε, δ are infinitesimals, b, c are hyperreal numbers that are finite but not infinitesimal, and H, K are infinite hyperreals

(i) Real numbers.

The only infinitesimal real number is 0.

Every real number is finite.

(ii) Negatives:

$-\varepsilon$ is infinitesimal

$-b$ is finite but not infinitesimal

$-H$ is infinite

(iii) Reciprocals:

If $\varepsilon \neq 0$, then $\frac{1}{\varepsilon}$ is infinite

$\frac{1}{b}$ is finite but not infinitesimal

$\frac{1}{H}$ is infinitesimal.

(iv) Sums

$\varepsilon + \delta$ is infinitesimal

$b + \varepsilon$ is finite but not infinitesimal

$b + c$ is finite (possibly infinitesimal)

$H + \varepsilon$ and $H + b$ are infinite

27 feb

(v) <u>products</u>

$\delta \cdot \varepsilon$ and $b \cdot \varepsilon$ are infinitesimal

$b \cdot c$ is finite but not infinitesimal

$H \cdot b$ and $H \cdot K$ are infinite

(vi) <u>quotients</u> [redacted]

$\frac{\varepsilon}{b}$, $\frac{\varepsilon}{H}$, and $\frac{b}{H}$ are infinitesimal

$\frac{b}{c}$ is finite but not infinitesimal

$\frac{b}{\varepsilon}$, $\frac{H}{\varepsilon}$, and $\frac{H}{b}$ are infinite, provided $\varepsilon \neq 0$.

(vii) <u>roots</u> (here n is a standard positive integer)

if $\varepsilon > 0$, then $\sqrt[n]{\varepsilon}$ is infinitesimal

if $b > 0$ then $\sqrt[n]{b}$ is finite but not infinitesimal

if $H > 0$, then $\sqrt[n]{H}$ is infinite.

Notice that we have given <u>no rules</u> for the following combinations

can be either infinitesimal, finite, or infinite
$$\left\{ \begin{array}{l} \dfrac{\varepsilon}{\delta} \\[2ex] \dfrac{H}{K} \\[2ex] H\varepsilon \\ H + K \end{array} \right.$$

Examples 1. ratio of two infinitesimals.

$\dfrac{\varepsilon^2}{\varepsilon}$ is infinitesimal (equal to ε).

$\dfrac{\varepsilon}{\varepsilon}$ is finite but not infinitesimal (equal to 1)

$\dfrac{\varepsilon}{\varepsilon^2}$ is infinite (equal to $\dfrac{1}{\varepsilon}$).

ratio of appreciable numbers is appreciable:

Example 2 Consider $\dfrac{6-3\varepsilon}{c+2\delta}$

prove it is finite but not infinitesimal

Ex 3 The quotient

$$\frac{5\varepsilon^4 - 8\varepsilon^3 + \varepsilon^2}{3\varepsilon}$$

is infinitesimal, provided $\varepsilon \neq 0$.

Ex 4 If $\varepsilon \neq 0$ then

$$\frac{3\varepsilon^3 + \varepsilon^2 - 6\varepsilon}{2\varepsilon^2 + \varepsilon}$$

is finite but not infinitesimal

בס"ד

27 feb

<u>Ex 5</u> If $\varepsilon \neq 0$ then

$$\frac{\varepsilon^4 - \varepsilon^3 + 2\varepsilon^2}{5\varepsilon^4 + \varepsilon^3}$$

is <u>infinite</u> (first argue that denom $\neq 0$).

<u>Ex 6</u>

$$\frac{2H^2 + H}{H^2 - H + 2}$$

is finite but not infinitesimal.

$$\frac{2 + \frac{1}{H}}{1 - \frac{1}{H} + \frac{2}{H^2}}$$

<u>Theorem</u> (i) every hyperreal number which is between two infinitesimals, is infinitesimal

(ii) every hyperreal number which is between two finite hyperreal numbers, is finite

(iii) every hyperreal number which is <u>greater</u> than some positive infinite number, is positive infinite

(iv) every hyperreal number which is less than some negative infinite number, is negative infinite.

Example 7 If H and K are positive infinite, then $H+K$ is positive infinite. This is because $H+K > H$.

Ex 8 If H is positive infinite, then

$$\sqrt{H+1} - \sqrt{H-1}$$

is infinitesimal.

Proof.

$$\sqrt{H+1} - \sqrt{H-1} = \frac{\left(\sqrt{H+1} - \sqrt{H-1}\right)\left(\sqrt{H+1} + \sqrt{H-1}\right)}{\sqrt{H+1} + \sqrt{H-1}}$$

$$= \frac{H+1 - (H-1)}{\sqrt{H+1} + \sqrt{H-1}}$$

$$= \frac{2}{\sqrt{H+1} + \sqrt{H-1}} = \frac{\text{finite}}{\text{infinite}} = \text{infinitesimal}$$