(3)

ג' אדר

6 march

Calculus $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{R}^*$

NB Goal today: derivative

Def. Two hyperreal numbers b and c are said to be infinitely close to each other, written

$$b \approx c$$

if their diff. $b-c$ is infinitesimal.

Notation $b \not\approx c$ means that b is not infinitely close to c.

~~Def~~ Rq (1) If ε is infinitesimal then $b \approx b+\varepsilon$.

(2) b is infinitesimal iff $b \approx 0$.

(3) if b and c are real and b is inf. close to c, then $b = c$.

Theorem 1 (i) $a \approx a$

(ii) if $a \approx b$ ~~then~~ $b \approx a$ ~~then~~.

(iii) $a \approx b$ + $b \approx c$ $\Rightarrow$ $a \approx c$

Thm 2. Assume $a \approx b$. Then

(i) if a is infinitesimal, then so is b

(ii) if a is finite, so is b

(iii) if a is infinite, so is b.

Standard part (principle) עיקרון

Every finite hyperreal number is infin close to exactly one real number.

Definition Let b be a finite hyperreal. The standard part of b, denoted

$$st(b)$$

is the real num which is inf. close to b.

Remark If b real then $st(b) = b$.

Theorem. Let a and b be finite hyperreal numbers. Then

(i) $st(-a) = -st(a)$

(ii) $st(a+b) = st\, a + st\, b$

(iii) $st(a-b) = st(a) - st(b)$

(iv) $st(ab) = st(a) \cdot st(b)$.

(v) If $\boxed{st(b) \neq 0}$ then $st\left(\frac{a}{b}\right) = \frac{st\, a}{st(b)}$

(vi) $st(a^n) = (st(a))^n$

(vii) If $a \geq 0$ then $st\left(\sqrt[n]{a}\right) = \sqrt[n]{st(a)}$.

(viii) If $a \leq b$ then $st(a) \leq st(b)$.

Example proof of (iv).

write $a = r + \varepsilon$, $b = s + \delta$

infinitesimal

Then $ab = (r + \varepsilon)(s + \delta)$

$= rs + r\varepsilon + s\varepsilon + \varepsilon\delta \approx rs$

Hence $st(ab) = rs = st(a)\ st(b)$.

Notation Δx, Δy for infinitesimals. increment, תוס'

Example 1 When Δx is infinitesimal and x real, compute st of

$$st = 3x^2 + 3x\,\Delta x + \Delta x^2 \ \vdots \ = 3x^2$$

Ex 2 Let $st(c) = 4$ and $c \neq 4$.

Calculate $st\left(\frac{c^2 + 2c - 24}{c^2 - 16}\right)$.

NB cannot apply $\frac{st(a)}{st(b)}$

factor $c-4$ to get $\frac{10}{8}$

Ex 3. Let H pos. infinite compute $\boxed{st}$ of

$$c = \frac{2H^3 + 5H^2 - 3H}{7H^3 - 2H^2 + 4H}.$$

in 3 stages: calc with hyperreals, with st parts, with reals.

$$st(c) = \frac{2}{7}$$

Ex 4. Let $\varepsilon \not\approx 0$. ~~Calc~~ but. $\varepsilon \neq 0$.

Find st of

$$b = \frac{\varepsilon}{5 - \sqrt{25+\varepsilon}}.$$

$$b = \frac{\varepsilon}{5 - \sqrt{\ }} \frac{(5 + \sqrt{\ })}{(5 + \sqrt{\ })}$$

$$= \dots -10$$

Ex 5. Consider hyperreal

$$\frac{3+\varepsilon}{4\varepsilon + \varepsilon^2}$$

Then $st(3+\varepsilon) = 3$, $st(4\varepsilon + \varepsilon^2) = 0$, but

st (quotient) is undefined

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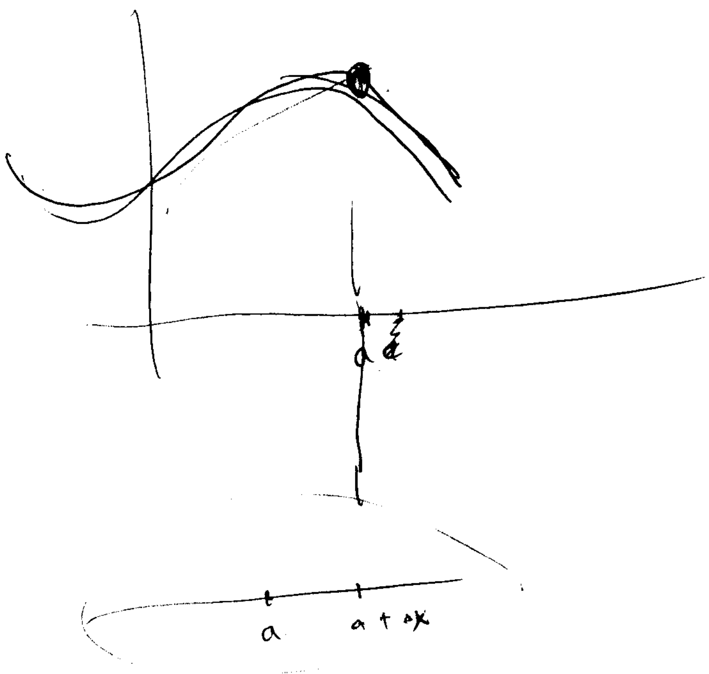
Differentiation

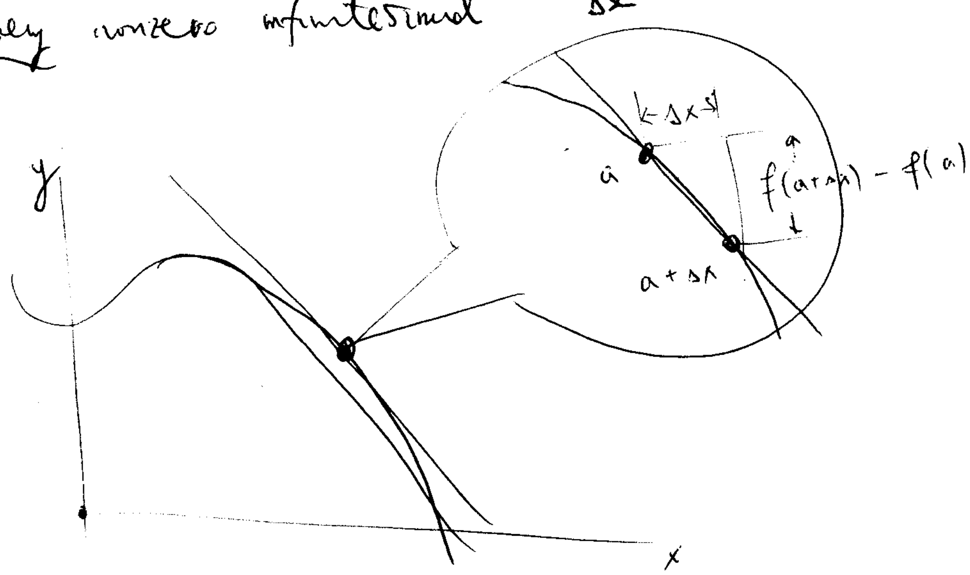


$$f(a+\Delta x) - f(a)$$

__Definition__ A real number s is said to be the __slope__ of f at a if

$$s = st\left(\frac{f(a+\Delta x) - f(a)}{\Delta x}\right)$$

for every nonzero infinitesimal Δx



The slope does not always exist. Here ~~are the~~ is the list of all 2 possibilities:

(1) The slope of f at a exists ~~and~~ if the ratio

$$\frac{f(a+\Delta x) - f(a)}{\Delta x}$$

is finite and has the same st. pt ~~reg~~ for all infinitesimal $\Delta x \neq 0$. Then the slope ~~st~~ has the value

$$S = st\left(\frac{f(a+\Delta x) - f(a)}{\Delta x}\right)$$

(2) The slope of f at a can fail to exist in any of four ways:

(a) $f(a)$ undefined

(b) $f(a+\Delta x)$ undefined for some inf $\Delta x \neq 0$

(c) the term ~~st pt~~ $\frac{f(a+\Delta x) - f(a)}{\Delta x}$ is infinite for some infinitesimal $\Delta x \neq 0$

(d) the term $\frac{f(a+\Delta x) - f(a)}{\Delta x}$ has different standard parts for different infinitesimals Δx.

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Def. Let f be a real function of one real variable. The derivative of f is the new function f' whose value at x is the slope of f at x. In symbols,

$$f'(x) = st\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)$$

whenever the slope exists.

Definition. In Equation $y = f(x)$,
y is the dependent variable
x is the independent variable.

When $y = f(x)$, we introduce a new independent variable

Δx

and a new dependent variable Δy, by eq

$$\Delta y = f(x+\Delta x) - f(x)$$

Δy called the y-increment 'תוס

$y' = f'(x)$.

Thus can write $y' = st\left(\frac{\Delta y}{\Delta x}\right)$.

Ex Find derivative of $f(x) = x^3$. (pay attn to 3 stages)

Start with eq $y = x^3$.

First we calculate $\frac{\Delta y}{\Delta x}$

$$y = x^3$$

$$y + \Delta y = (x + \Delta x)^3$$

$$\Delta y = (x + \Delta x)^3 - x^3$$

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^3 - x^3}{\Delta x}$$

Now simplify:

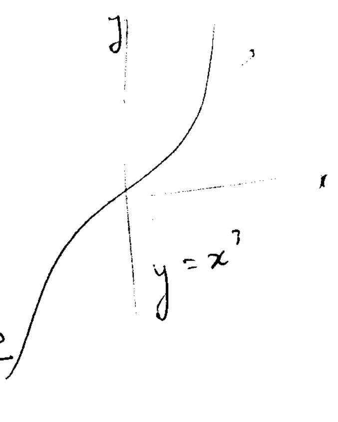
stage 1 $\quad \frac{\Delta y}{\Delta x} = \ldots \quad 3x^2 + 3x\,\Delta x + \Delta x^2.$

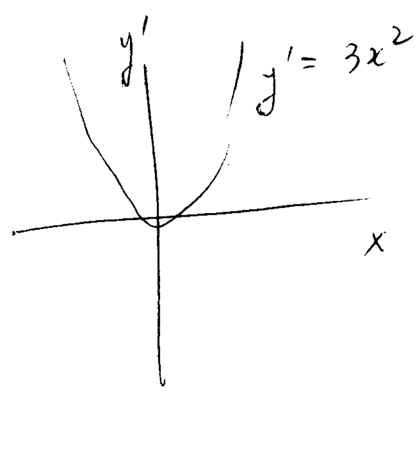
stage 2 $\quad st\left(\frac{\Delta y}{\Delta x}\right) = \ldots = 3x^2$

Thus if $f(x) = x^3$

then $f'(x) = 3x^2$.

Domain: whole real line





↓ 24 Oct '13

<u>Ex 2</u>. Find $f'(x)$ given $f(x) = \sqrt{x}$ 6 march

<u>Case 1</u> $x < 0$. Since $\sqrt{x}$ not defined, $f'(x)$ does not exist

<u>Case 2</u> $x = 0$. when $\Delta x < 0$, Δy not defined.

When $\Delta x > 0$, as $\dfrac{\sqrt{\Delta x}}{\Delta x} = \dfrac{1}{\sqrt{\Delta x}}$ ~~undefined~~ is infinite.
$\Rightarrow f'(x) \nexists$.

<u>Case 3</u> $x > 0$.

$y = \sqrt{x}$. $\dfrac{\Delta y}{\Delta x} = \dfrac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$

$$= \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(\frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}\right)$$

stage 1 $= \dfrac{1}{st\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}$

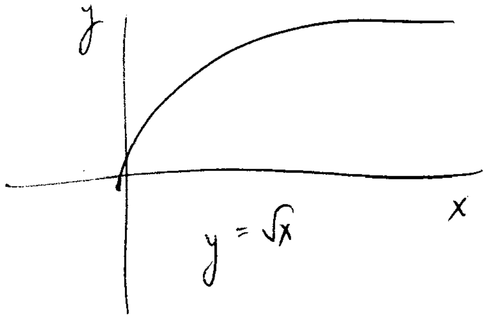
stage 2 $= \dfrac{1}{st\left(\sqrt{x+\Delta x}\right) + st\left(\sqrt{x}\right)}$

stage 3 $= \dfrac{1}{\sqrt{x} + \sqrt{x}}$

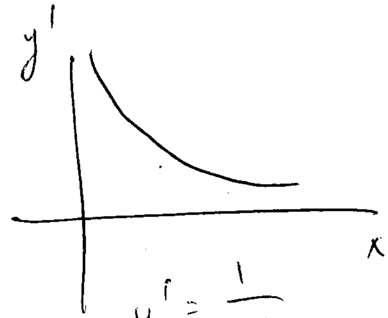
$$= \frac{1}{2\sqrt{x}}$$

Thus when $x > 0$, $f'(x) = \dfrac{1}{2\sqrt{x}}$

Domain of f' is $x > 0$.



$y = \sqrt{x}$



$y' = \frac{1}{2\sqrt{x}}$

<u>Example 3</u>. $f(x) = \frac{1}{x}$. domain $\{x \neq 0\}$

<u>case 1</u> $x = 0$. $\frac{1}{x}$ undefined, so $f(x)$ undefined.

<u>case 2</u> $x \neq 0$.

$$y = \frac{1}{x}$$

Stage 1

$$y + \Delta y = \frac{1}{x + \Delta x}$$

$$\Delta y = \frac{1}{x + \Delta x} - \frac{1}{x}$$

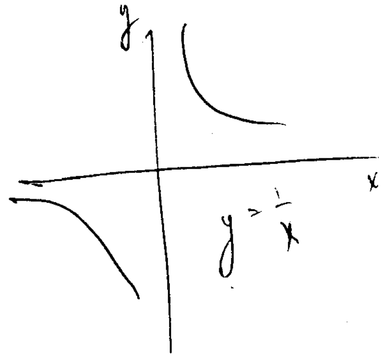
$$\frac{\Delta y}{\Delta x} = \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x}$$

$$= \frac{-1}{x(x + \Delta x)}$$

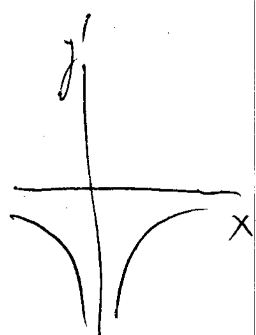
Stage 2 st $= -\frac{1}{x^2}$

$$f'(x) = \frac{-1}{x^2}$$

Domain: $\{x \neq 0\}$



$y = \frac{1}{x}$



$y' = \frac{-1}{x^2}$