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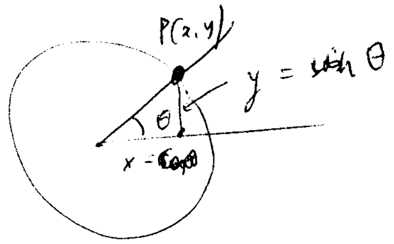
3rd 02
10 april

Calculus

Transcendental functions

Recall



$$\sin^2\theta + \cos^2\theta = 1 \quad \forall\theta$$

$$\frac{d(\sin\theta)}{d\theta} = \cos\theta$$

$$\frac{d(\cos\theta)}{d\theta} = -\sin\theta$$

Example 1 Differentiate $y = \sin^2\theta$

Let $u = \sin\theta$

$y = u^2$

$$\frac{dy}{d\theta} = 2u\frac{du}{d\theta} = 2\sin\theta\cos\theta$$

Ex 2 Differentiate $y = \sin\theta\,(1 - \cos\theta)$

Let $u = \sin\theta,\quad v = 1 - \cos\theta$

$y = u \cdot v$

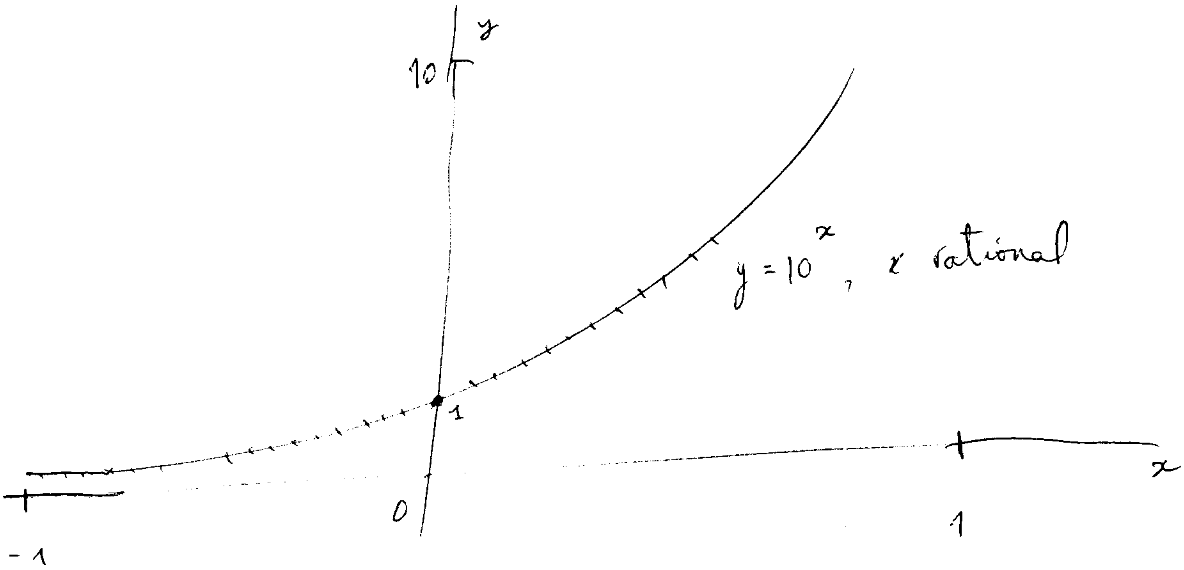
$$\frac{dy}{d\theta} = u\frac{dv}{d\theta} + v\frac{du}{d\theta} = \sin\theta\left(-(-\sin\theta)\right) + (1-\cos\theta)\cos\theta$$

$$= \sin^2\theta + \cos\theta - \cos^2\theta$$

Exponential function. Given positive real b and rational m/n, rational powers $b^{m/n} = \sqrt[n]{b^m}$.

The negative power $b^{-m/n}$ is

$$b^{-m/n} = \frac{1}{b^{m/n}}$$



By connecting the dots with a smooth curve, we get $y = 10^x$ for all real x. Rules:

$$10^{a+b} = 10^a \cdot 10^b$$

$$10^{a \cdot b} = (10^a)^b$$

Derivative of 10^x is a constant times 10^x, approx

$$\frac{d(10^x)}{dx} \sim (2.303)\ 10^x$$

Indeed, let Δx infinitesimal. Then

$$\frac{d(10^x)}{dx} = st\left(\frac{10^{x+\Delta x} - 10^x}{\Delta x}\right) = \underbrace{st\left(\frac{10^{\Delta x} - 1}{\Delta x}\right)}_{\sim 2.303} \cdot 10^x$$

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Starting with positive real b instead of 10,

~~get~~ $y = b^x$, exponential function with base 10.

$$\frac{d(b^x)}{dx} = \ldots \quad st\left(\frac{b^{\Delta x} - 1}{\Delta x}\right) b^x$$

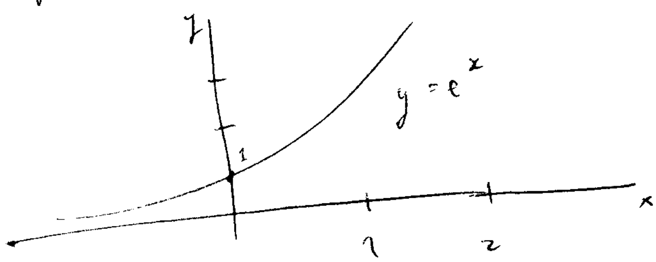
<u>Definition</u>. Real number e is the number such that the derivative of e^x is e^x.

$$\frac{d(e^x)}{dx} = e^x$$

$$st\left(\frac{e^{\Delta x} - 1}{\Delta x}\right) = 1$$

$$e \sim 2.71828$$

The function $y = e^x$ is called the exponential fun.



<u>Example</u> Find derv of $y = x^2 e^x$

$$\frac{dy}{dx} = x^2 \frac{d(e^x)}{dx} + e^x \frac{d(x^2)}{dx} = x^2 e^x + 2x e^x$$

14 nov

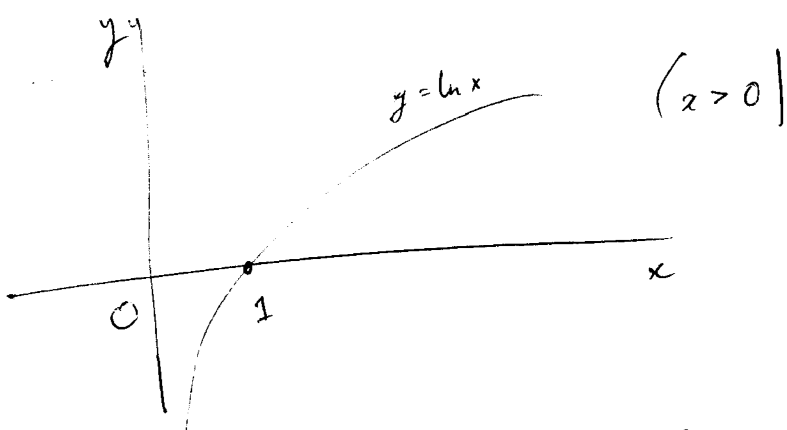
Definition The inverse of the exponential function $x = e^y$ is the natural logarithm function, written

$$y = \log x \qquad (y = \ln x).$$

Hence $$e^{\log a} = a, \qquad \log(e^a) = a.$$

Simplest values

$$\log\left(\frac{1}{e}\right) = -1$$
$$\ln(1) = 0$$
$$\ln(e) = 1$$



Rules $$\log(ab) = \log a + \log b$$
$$\log(a^b) = b \log a$$

Theorem $$\frac{d(\log x)}{dx} = \frac{1}{x} \qquad (x > 0)$$

Proof $y = \log x$, $x = e^y$, $\frac{dx}{dy} = e^y$, $\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}$

Example Diff $\frac{1}{\log x}$

$$\frac{dy}{dx} = \frac{-1}{(\log x)^2}\,\frac{d(\log x)}{dx} = -\frac{1}{x(\log x)^2}$$

Chain rule

כלל שרשרת

Definition

$x = f(t)$, $y = G(x)$

Then

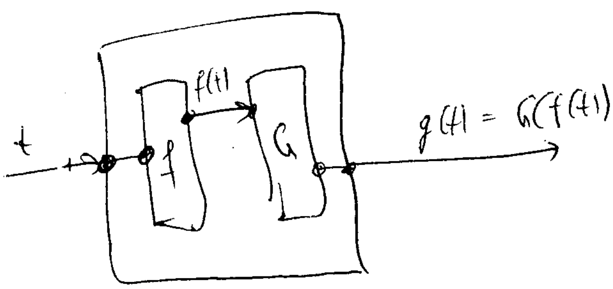
$y = g(t)$ where

$$g(t) = G(f(t)).$$

is called the composition

הרכבה

$$g = G \circ f$$



Compos. $g = G \circ f$

<u>Chain rule</u> (with function) let f, G be two real functions and define a new real func g by the rule

$$g(t) = G\,(f(t)).$$

At any value of t where the derivatives $f'(t)$ and $G'(f(t))$ exist, also $g'(t)$ exists and

$$g'(t) = G'\,(f(t))\,f'(t)$$

<u>Proof.</u> Let $x = f(t)$

$$y = g(t)$$

$$y = G(x)$$

Let $\Delta t \neq 0$ infinitesimal.

let Δx and Δy be corr. increments.

Then Δx is infinitesimal by the increment theorem.

Applying increment theorem to $y = G(x)$, get

$$\Delta y = G'(x)\,\Delta x + \varepsilon\,\Delta x$$

divide by Δt get

$$\frac{\Delta y}{\Delta t} = G'(x)\,\frac{\Delta x}{\Delta t} + \varepsilon\,\frac{\Delta x}{\Delta t}$$

standard part gives

$$st\left(\frac{\Delta y}{\Delta t}\right) = G'(x)\;st\left(\frac{\Delta x}{\Delta t}\right) + 0$$

$$g'(t) = G'(x)\,f'(t) \;\boxtimes$$

Example 1 Diff $g(t) = \log(\sin t)$

$$f(t) = \sin t$$
$$G(x) = \log x$$

$$f'(t) = \cos t$$
$$G'(x) = \frac{1}{x}$$

Thus $$g'(t) = \frac{1}{\sin t} \cdot \cos t = \frac{\cos t}{\sin t}$$

Ex 2 Diff. $g(t) = \sqrt{3t+1}$.

Here $g(t) = G(f(t))$

$$f(t) = 3t+1$$
$$G(x) = \sqrt{x}$$

$$f'(t) = 3$$
$$G'(x) = \frac{1}{2} x^{-\frac{1}{2}}$$

$$g'(t) = \frac{1}{2}(3t+1)^{-1/2} \cdot 3 = \frac{3}{2\sqrt{3t+1}}$$

Chain rule with dependent variables

$x = f(t)$, $y = g(t) = G(x)$.

Assume $f'(t)$ and $G'(x)$ exist. Then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$$dy = \frac{dy}{dx}dx$$

Ex 1 $\quad x = \sin t, \quad y = \log x$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{x}\cos t = \frac{\cos t}{\sin t}$$

Alternatively

$dx = \cos t \; dt,$

$$dy = \frac{1}{x}\,dx = \frac{1}{x}\cos t \; dt = \frac{\cos t}{\sin t}\,dt$$

Ex 2. $\quad x = 3t+1, \quad y = \sqrt{x}$

$$\frac{dx}{dt} = 3, \qquad \frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{3}{2}x^{-1/2} = \frac{3}{2}(3t+1)^{-1/2}$$

Power rule Let r be a rational number.

Let u depend on x. If $u > 0$ and $\frac{du}{dx}$ exists,

then
$$\frac{d(u^r)}{dx} = r\,u^{r-1}\frac{du}{dx}$$

Proof By chain rule: let $y = u^r$

compute $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = r\,u^{r-1}\frac{du}{dx} \quad \square$$

Ex. Find $\frac{dy}{dt}$ when $y = (5t^2 - 2)^{1/4}$

Ex. $\frac{dy}{dx}$ when $y = \sqrt{\sin(4x+1) + \cos(4x-1)}$