7

בס"ד

april

Ex. A particle moves acc. to param. eq

$x = t^3 - t$

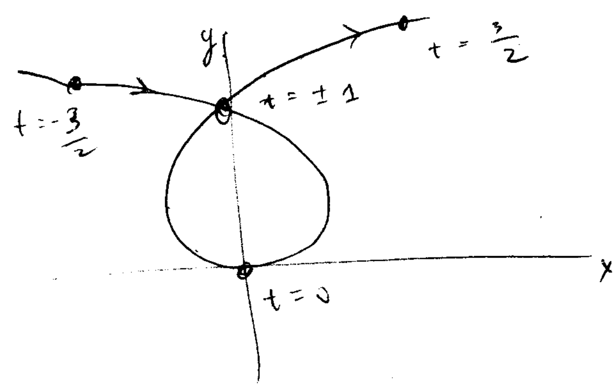
$y = t^2$

Find slope of path.

$\frac{dx}{dt} = 3t^2 - 1$

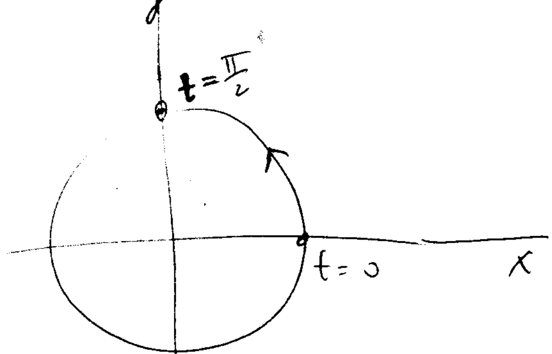
$\frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2-1}$, $t \neq \pm\sqrt{\frac{1}{3}}$.



Ex. Particle $x = \cos t$
$y = \sin t$

Find slope at time t.



$x = \cos t$, $y = \sin t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{\cos t}{\sin t}$

$\frac{dy}{dx} = -\frac{x}{y}$

↓ 21 nov

Def The second derivative of a real function f is the derivative of the derivative of f, and is denoted f''.
The third deriv. is the deriv. of the second, and denoted f''', or $f^{(3)}$. In general, the n-th deriv. is denoted $f^{(n)}$

If y depends on x, then the second differential

$$d^2y$$

is defined to be

$$d^2y = f''(x)\,dx^2$$

And similarly

$$d^ny = f^{(n)}(x)\,dx^n$$

Here dx^2 means $(dx)^2$ and $dx^n = (dx)^n$.

Thus have alternative notation

$$\frac{d^2y}{dx^2} = f''(x)$$

$$\frac{d^ny}{dx^n} = f^{(n)}(x)$$

~~Motivation Consider $dy = f'(x)\,dx$~~
~~Formally apply product rule~~
~~$d(dy) = f''(x)\,dx\,dx$~~

3"02

~~3~~ -

10 april

Ex acceleration

$$a = \frac{dv}{dt}$$

$$v = \frac{ds}{dt}$$

$$a = \frac{d^2s}{dt^2}$$

Ex 1 Ball thrown up with initial velocity 6

moves acc to equation

$$y = 6t - \tfrac{1}{2}gt^2 \qquad g = 9.8 \text{ m/sec}^2$$

Then $v = 6 - gt$

$a = -g$

Ex 2 Find second der of $y = \sin(2\theta)$

Ex poly deg n

then k-th deriv is deg $n-k$ for $k \le n$.

n-the der is constant

$(n+1)$th der is zero. e.g.

$$y = 3x^5 - 10x^4 + x^2 - 7x + 4$$

$\vdots$

$$\frac{d^6 y}{dx^6} = 0.$$

Implicit functions
פונקציה סתומה

<u>Definition</u> We say that y is an implicit fn of x if we are given an equation

$$\sigma(x,y) = \tau(x,y)$$

which determines y as a function of x.

<u>Example</u> $x + xy = 2y$

<u>Implicit differention</u> is a way of finding the derv. of y wrt x without actually solving for y as a func of x. Assume $\frac{dy}{dx}$ exists.

Procedure is in 2 steps:

(1) differentiate both sides of equation to get a new eq

$$\frac{d(\sigma(x,y))}{dx} = \frac{d(\tau(x,y))}{dx}$$

this step uses <u>chain rule</u>

(2) Solve the new equation from step (1) for $\frac{dy}{dx}$.

The answer will usually involve <u>both x and y</u>.

<u>Ex</u> $x + xy = 2y$

Step 1

$$\frac{d(x+xy)}{dx} = \frac{d(2y)}{dx}$$

$$\frac{d(x+xy)}{dx} = \frac{dx + d(xy)}{dx} = \frac{dx + dx \cdot y + x\,dy}{dx}$$

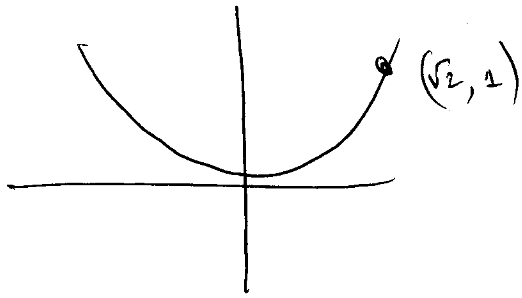
$$= 1 + y + \frac{dy}{dx}$$

$$\frac{d(2y)}{dx} = 2\frac{dy}{dx}$$

<u>Step 2</u> $1 + x\frac{dy}{dx} + y = 2\frac{dy}{dx}$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1+y}{2-x}$$



<u>Ex 2</u> given $y + \sqrt{y} = x^2$, find $\frac{dy}{dx}$.

⋮

$$\frac{dy}{dx} = \frac{2x}{1 + \frac{1}{2}y^{-\frac{1}{2}}}$$

At the point $(\sqrt{2}, 1)$ $\quad \frac{dy}{dx} = \frac{4\sqrt{2}}{3}$

<u>Ex 3</u> $x^2 - 2y^2 = 4, \quad y \le 0$

Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{x}{2y}$$

$$y = -\sqrt{\frac{x^2-4}{2}}$$

<u>Ex 4</u> $x^2 + y^2 = 1$ <u>NB</u> does <u>not</u> define y as a function of x. Still can find $\frac{dy}{dx}$.

↓ 26 nov

$$\frac{dy}{dx} = -\frac{x}{y}$$

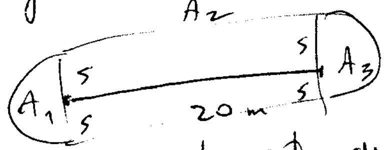
<u>Find slope of tangent line</u> at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(0,-1)$, $(-1, 0)$.

<u>Ex.</u> Slope of tangent line to $x^5y^3 + xy^6 = y + 1$ at points $(1,1)$, $(1,-1)$, $(0,-1)$.

Area, word problems

what's involved is problem solving

Fire starts along a straight line segment 20 m and expands in all directions at rate ② m/sec.



Find burnt out area as function of time.

Step 1 A = total area

A_1 = area left semicircle

A_2 = area of central rectangle

A_3 = area of right semicircle

s = distance of spread of fire

$s = 2t$.

Step 2

$$A_1 = \frac{1}{2}\pi s^2$$

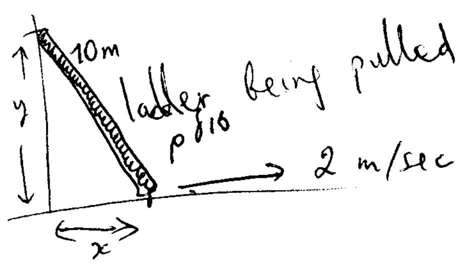
$$A_2 = 20(2s)$$

$$A_3 = \frac{1}{2}\pi s^2$$

$$A = A_1 + A_2 + A_3$$

Step 3 $A = 2\pi t^2 + 80t + 2\pi t^2 = 4\pi t^2 + 80t$

Related rates problem



Find the rate at which top end is sliding when bottom end is 5 m from wall.

17 april

<u>Step 1</u> t = time

x = distance of bottom end to wall

y = height of top end above floor.

<u>Step 2</u> $\frac{dx}{dt} = 2$, $x^2 + y^2 = 10^2$

<u>Step 3</u> differentiate

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$4x + 2y \frac{dy}{dt} = 0$$

<u>Step 4</u> set $x = 5$.

Solve for $\frac{dy}{dt}$

First find $y = \sqrt{100 - x^2} = \sqrt{75}$

$$\frac{dy}{dt} = -\frac{4 \cdot 5}{2 \cdot \sqrt{75}} = -\frac{2}{\sqrt{3}} \text{ m/sec.}$$

<u>Formulas used in word problems</u>

area of triangle $A = \frac{1}{2} bh$

area of circle rad r is πr^2

circumference of היקף $2\pi r$

Volume of solid box with sides a, b, c $V = abc$

sphere of radius r. $V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$

right circular cylinder $V = \pi r^2 h$, $A = 2\pi r h$ (w/o caps)

Limits

Def L is the limit of $f(x)$ as x approaches c iff whenever x is infinitely close but not equal c, the value $f(x)$ is infinitely close to L.

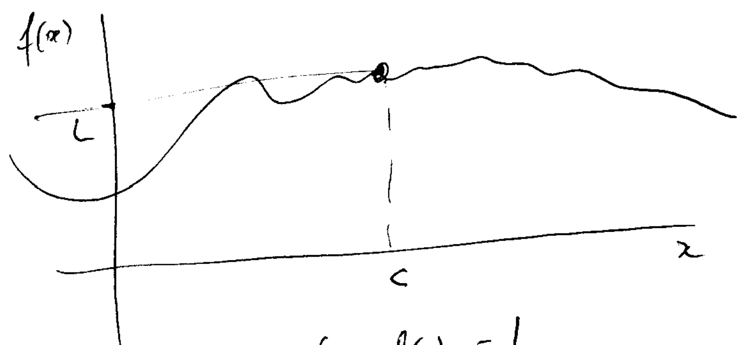
In symbols,

$$\lim_{x \to c} f(x) = L$$

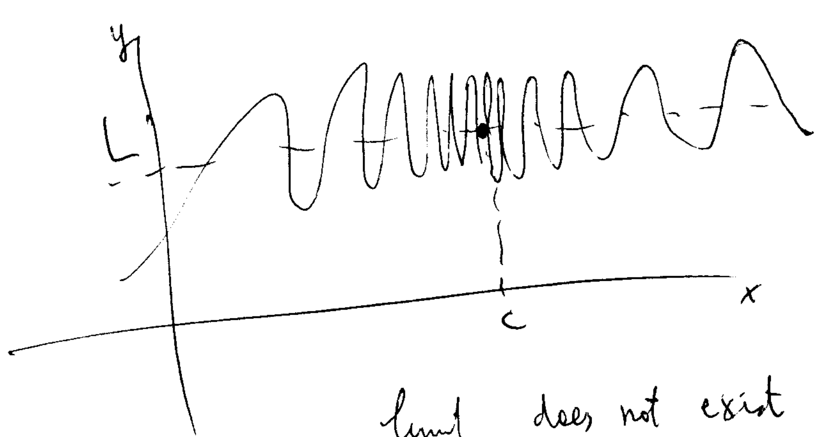
if whenever $x \approx c$ but $x \neq c$, one has $f(x) \approx L$.

If $\nexists$ such L, we say limit does not exist.

NB limit $\lim_{x \to c} f(x)$ depends only on values of x infinitely close to but not equal to c.
The value $f(c)$ itself has no influence at all on limit.



$$\lim_{x \to c} f(x) = L$$



limit does not exist

Comment on the function $\sin \frac{1}{x}$

<u>Theorem</u> The slope of f at a is given by the limit

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(x)}{\Delta x}$$

Alternatively $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

<u>Example</u> evaluate $$\lim_{\Delta x \to 0} \frac{(3+\Delta x)^2 - 9}{\Delta x}$$

Let $F(x) = x^2$

<u>Ex</u> $$\lim_{x \to 2} \frac{x^2 + 3x - 10}{x^2 - 4}$$

<u>Ex</u> find $$\lim_{x \to 0} \frac{\frac{2}{x} + 3}{\frac{3}{x} - 1}$$

<u>Step 1</u> take $x \approx 0$ but $x \neq 0$

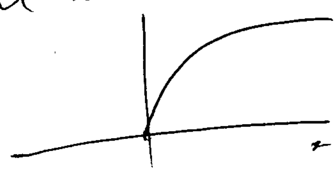
Then get ...... $\frac{2}{3}$

<u>Ex</u> find $$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$$

Table of comparison of standard part rules and limit rules

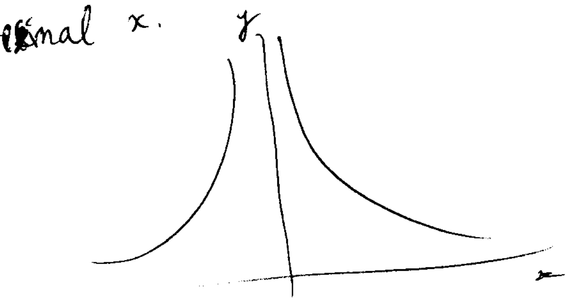
standard part rule	limit rule
$st(kb) = k\,st(b)$, k real	$\lim_{x \to c} k f(x) = k \lim_{x \to c} f(x)$
$st(a+b) = st(a) + st(b)$	$\lim_{x \to c} \left(f(x) + g(x)\right) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
$st(ab) = st(a)\,st(b)$	$\lim_{x \to c} f(x)\,g(x) =$
$st\left(\frac{a}{b}\right) =$ if...	$\lim_{x \to c} \frac{f(x)}{g(x)} =$ if ...
$st(\sqrt[n]{a}) = \sqrt[n]{st(a)}$ if $a > 0$	$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$ if $\lim_{x \to c} f(x) > 0$ (otherwise root may be undefined)

Ex 7 $\lim_{x \to 0} \sqrt{x}$ does not exist because $\sqrt{x}$ is undefined for negative infinitesimal x.



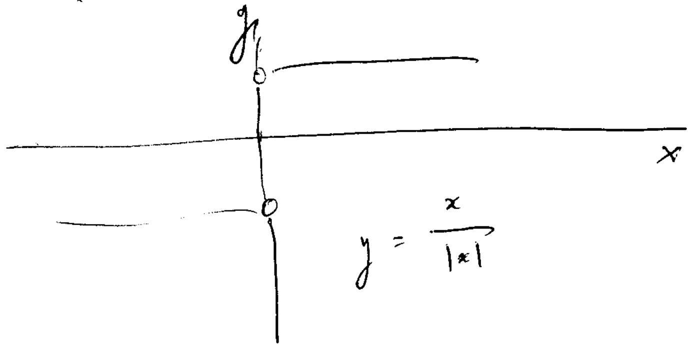
$y = \sqrt{x}$

Ex 8 $\lim_{x \to 0} \frac{1}{x^2}$ does not exist because $\frac{1}{x^2}$ is infinite for infinitesimal x.



$y = \frac{1}{x^2}$

Ex 3 $\lim_{x \to 0} \frac{x}{|x|}$ does not exist because $st\left(\frac{x}{|x|}\right) = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$



$y = \frac{x}{|x|}$

One-sided limits

We say that $\lim_{x \to c^+} f(x) = L$ if whenever $x > c$ and $x \approx c$, we have $f(x) \approx L$. Similarly $\lim_{x \to c^-} f(x) = L$

Theorem A limit has value L, $\lim f(x) = L$ iff both one-sided limits exist and are equal to L. Proof either $x > c$ or $x < c$.

Ex $\lim_{x \to 0^+} \sqrt{x} = 0$, $\lim_{x \to 0}$ does not exist.