8

One-sided limits

We say that

$$\lim_{x \to c^+} f(x) = L$$

if whenever $x > c$ and $x \approx c$, $f(x) \approx L$.

$$\lim_{x \to c^-} f(x) = L$$

means that whenever $x < c$ and $x \approx c$, $f(x) \approx L$. These two kinds of limits, shown in Figure 3.3.4, are called the *limit from the right* and the *limit from the left*.

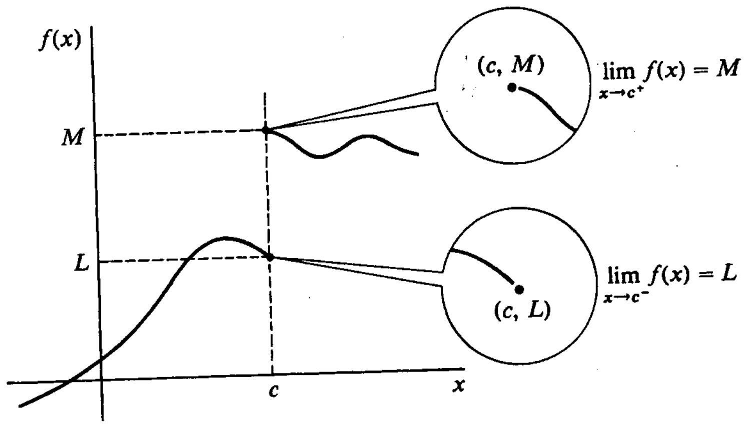


Figure 3.3.4 One-sided limits.

THEOREM 2

A limit has value L,

$$\lim_{x \to c} f(c) = L,$$

if and only if both one-sided limits exist and are equal to L,

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$$

PROOF If $\lim_{x \to c} f(x) = L$, it follows at once from the definition that both one-sided limits are L.

Assume that both one-sided limits are equal to L. Let $x \approx c$, but $x \neq c$. Then either $x < c$ or $x > c$. If $x < c$, then because $\lim_{x \to c^-} f(x) = L$, we have $f(x) \approx L$. On the other hand if $x > c$, then $\lim_{x \to c^+} f(x) = L$ gives $f(x) \approx L$. Thus in either case $f(x) \approx L$. This shows that $\lim_{x \to c} f(x) = L$.

When a limit does not exist, it is possible that neither one-sided limit exists, that just one of them exists, or that both one-sided limits exist but have different values.

EXAMPLE 7 (Continued) $\lim_{x \to 0^+} \sqrt{x} = 0$, and $\lim_{x \to 0^-} \sqrt{x}$ does not exist.

EXAMPLE 8 (Continued) Neither $\lim_{x \to 0^+} 1/x^2$ nor $\lim_{x \to 0^-} 1/x^2$ exists.

EXAMPLE 9 (Continued) $\lim_{x\to 0^+} x/|x| = 1,\qquad$ and $\qquad\lim_{x\to 0^-} x/|x| = -1.$

PROBLEMS FOR SECTION 3.3

In each problem below, determine whether or not the limit exists. When the limit exists, find its value.

1 $\lim_{t\to 4} 3t^2 + t + 1$

2 $\lim_{\Delta x\to -1} \dfrac{\Delta x^2 + 2\Delta x + 1}{\Delta x + 1}$

3 $\lim_{x\to c} \sqrt{c - x}$

4 $\lim_{y\to 0} \dfrac{1}{y^5}$

5 $\lim_{x\to 2} \dfrac{x}{x^2 - 4}$

6 $\lim_{x\to 2} \dfrac{x^2 - 4}{x - 2}$

7 $\lim_{v\to 8} \dfrac{\sqrt{8} - \sqrt{v}}{v - 8}$

8 $\lim_{x\to 0} \dfrac{\sqrt{x+1} - 1}{x}$

9 $\lim_{u\to 1} \dfrac{\sqrt[3]{u} - 1}{u - 1}$

10 $\lim_{t\to 0} \dfrac{t^3 - 2t^2 + 4}{3t^2 - 5t + 7}$

11 $\lim_{y\to 0} (\sqrt{1 + 1/y} - \sqrt{1/y}$

12 $\lim_{x\to 0} \dfrac{(a + x)^2 - a^2}{x}$

13 $\lim_{y\to -1} \dfrac{y^2 + 1}{y + 1}$

14 $\lim_{x\to 1} \dfrac{|x - 1|}{x - 1}$

15 $\lim_{x\to 1^+} \dfrac{|x - 1|}{x - 1}$

16 $\lim_{x\to c^-} \sqrt{c - x}$

17 $\lim_{z\to 1} \sqrt{z + \sqrt{z + \sqrt{z}}}$

18 $\lim_{x\to a} \sqrt{|a - x|}$

19 $\lim_{x\to 0^+} x\sqrt{1 + x^{-2}}$

20 $\lim_{x\to 0^-} x\sqrt{1 + x^{-2}}$

21 $\lim_{t\to 0} \dfrac{1 + 2t^{-1}}{3 - 4t^{-1}}$

22 $\lim_{x\to 0} \dfrac{3 + 4x^{-1} - 5x^{-2}}{6 - x^{-1} + 3x^{-2}}$

23 $\lim_{\Delta x\to 0} \dfrac{(x + \Delta x)^2 - x^2}{\Delta x}$

24 $\lim_{\Delta x\to 0} \dfrac{\dfrac{1}{x + \Delta x} - \dfrac{1}{x}}{\Delta x} \quad (x \neq 0)$

25 $\lim_{\Delta t\to 0} \dfrac{\sqrt{t + \Delta t} - \sqrt{t}}{\Delta t} \quad (t > 0)$

26 $\lim_{\Delta t\to 0} \dfrac{(t + \Delta t)^{1/5} - t^{1/5}}{\Delta t} \quad (t > 0)$

27 $\lim_{\Delta x\to 0} \dfrac{(x - \Delta x)^3 - x^3}{\Delta x}$

28 $\lim_{\Delta x\to 0} \dfrac{\dfrac{x + \Delta x}{x + \Delta x + 1} - \dfrac{x}{x + 1}}{\Delta x} \quad (x \neq -1)$

29 $\lim_{\Delta x\to 0^-} \dfrac{|(1 + \Delta x)^3 - (1 + \Delta x)|}{\Delta x}$

30 $\lim_{\Delta x\to 0^+} \dfrac{|(1 + \Delta x)^3 - (1 + \Delta x)|}{\Delta x}$

31 $\lim_{\Delta x\to 0^-} \dfrac{\sqrt{1 - (1 + \Delta x)^2}}{\Delta x}$

3.4 CONTINUITY

Intuitively, a curve $y = f(x)$ is continuous if it forms an unbroken line, that is, whenever x_1 is close to x_2, $f(x_1)$ is close to $f(x_2)$. To make this ~~intuitive idea~~ into a mathematical definition, we substitute "infinitely close" for "close."

Cauchy, Cours d'Analyse, 1821

2.2 On the continuity of functions.

[43] Among the objects related to the study of infinitely small quantities, we ought to include ideas about the continuity and the discontinuity of functions. In view of this, let us first consider functions of a single variable.

Let $f(x)$ be a function of the variable x, and suppose that for each value of x between two given limits, the function always takes a unique finite value. If, beginning with a value of x contained between these limits, we add to the variable x an infinitely small increment α, the function itself is incremented by the difference[6]

$$f(x+\alpha)-f(x),$$

which depends both on the new variable α and on the value of x. Given this, the function $f(x)$ is a *continuous* function of x between the assigned limits if, for each value of x between these limits, the numerical value of the difference

$$f(x+\alpha)-f(x)$$

Cauchy's definition of continuity: not an epsilon (or delta) in sight

decreases indefinitely with the numerical value of α. In other words, *the function $f(x)$ is continuous with respect to x between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself.*[7]

We also say that the function $f(x)$ is a continuous function of the variable x in a neighborhood of a particular value of the variable x whenever it is continuous between two limits of x that enclose that particular value, even if they are very close together.

Finally, whenever the function $f(x)$ ceases to be continuous in the neighborhood of a particular value of x, we say that it becomes discontinuous, and that there is *solution of continuity*[8] for this particular value.

[44] Having said this, it is easy to recognize the limits between which a given function of a variable x is continuous with respect to that variable. So, for example, the function $\sin x$, which takes a unique finite value for each particular value of the variable x, is continuous between any two limits of this variable, given that the numerical value of $\sin\left(\frac{1}{2}\alpha\right)$, and consequently that of the difference[9]

$$\sin(x+\alpha)-\sin x = 2\sin\left(\tfrac{1}{2}\alpha\right)\cos\left(x+\tfrac{1}{2}\alpha\right),$$

---

[6] Cauchy defines continuity only on the interior of a bounded interval, and for the whole interval, not just at a single point. See [Grabiner 2005, p. 87] for more on this point. This passage is also cited in [DSB Cauchy, p. 136].

[7] [Grattan-Guinness 1970b] has suggested that Cauchy "stole" this and other ideas from Bolzano's paper of 1817. See also [Freudenthal 1971b, Jahnke 2003, p. 161, Grabiner 2005, pp. 9–12].

[8] This word "solution" takes an old meaning here; it means that continuity dissolves or disappears.

[9] To verify this formula, let $u = x+\frac{1}{2}\alpha$ and $v=\frac{1}{2}\alpha$, then apply the usual formula for $\sin(a+b)$ to the expression $\sin(u+v)-\sin(u-v)$.

DEFINITION

*f is said to be **continuous** at a point c if :*

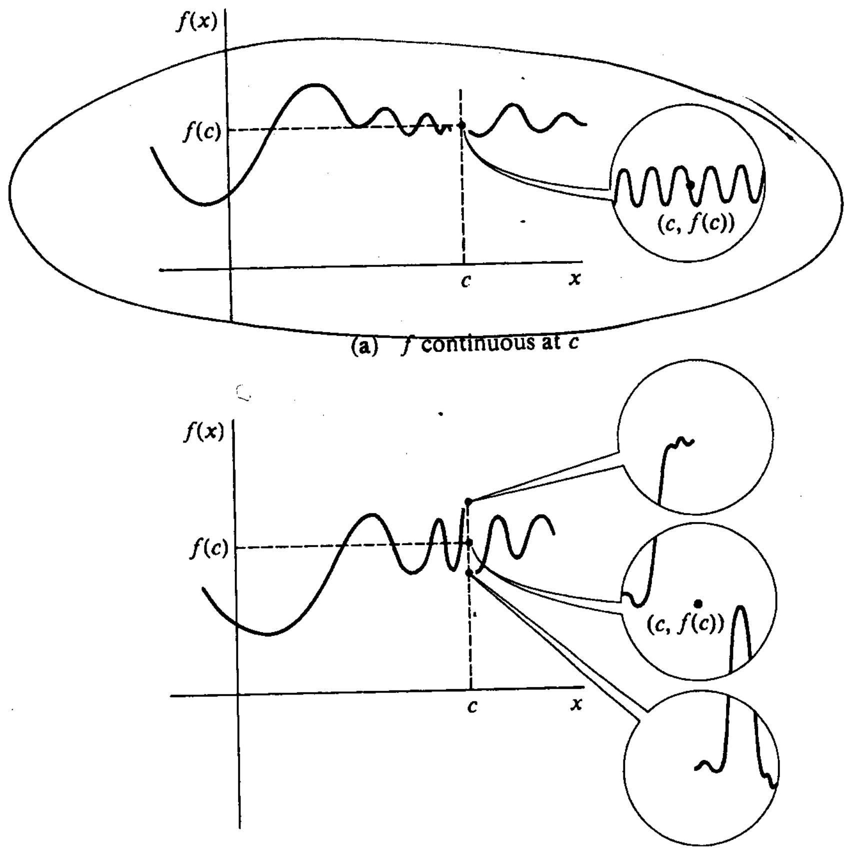
(i) *f is defined at c;*

(ii) *whenever x is infinitely close to c, $f(x)$ is infinitely close to $f(c)$.*

If f is not continuous ~~at c it is said to~~ *be discontinuous at c.*

When f is continuous at c, the entire part ~~of the curve~~ where $x \approx c$ will be visible in an infinitesimal microscope aimed at the point $(c, f(c))$, as shown in Figure 3.4.1(a). But if f is discontinuous at c, some values of $f(x)$ where $x \approx c$ will either be undefined or outside the range of vision of the microscope, as in Figure 3.4.1(b).

Continuity, like the derivative, can be expressed in terms of limits. Again the proof is immediate from the definitions.



(a) f continuous at c

Figure 3.4.1 (b) f discontinuous at c

THEOREM 1

f is continuous at c if and only if

$$\lim_{x \to c} f(x) = f(c).$$

As an application, we have a set of rules for combining continuous functions. They can be proved either by the corresponding rules for limits (Table 3.3.1 in Section 3.3) or by computing standard parts.

THEOREM 2

Suppose f and g are continuous at c.

(i) *For any constant k, the function $k \cdot f(x)$ is continuous at c.*
(ii) *$f(x) + g(x)$ is continuous at c.*
(iii) *$f(x) \cdot g(x)$ is continuous at c.*
(iv) *If $g(c) \neq 0$, then $f(x)/g(x)$ is continuous at c.*
(v) *If $f(c)$ is positive and n is an integer, then $\sqrt[n]{f(x)}$ is continuous at c.*

By repeated use of Theorem 2, we see that all of the following functions are continuous at c.

Every polynomial function.
Every rational function $f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials and $g(c) \neq 0$.
The functions $f(x) = x^r$, r rational and x positive.

Sometimes a function $f(x)$ will be undefined at a point $x = c$ while the limit

$$L = \lim_{x \to c} f(x)$$

exists. When this happens, we can make the function continuous at c by defining $f(c) = L$.

EXAMPLE 1 Let $f(x) = \dfrac{x^2 + x - 2}{x - 1}$.

At any point $c \neq 1$, f is continuous. But $f(1)$ is undefined so f is discontinuous at 1. However,

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 2)}{x - 1} = 3.$$

We can make f continuous at 1 by defining

$$f(x) = \begin{cases} \dfrac{x^2 + x - 2}{x - 1} & \text{if } x \neq 1, \\ 3 & \text{if } x = 1. \end{cases}$$

(See Figure 3.4.2.)

In terms of a dependent variable $y = f(x)$, the definition of continuity takes the following form, where $\Delta y = f(c + \Delta x) - f(c)$.

y is continuous at $x = c$ if:

(i) *y is defined at $x = c$.*
(ii) *Whenever Δx is infinitesimal, Δy is infinitesimal.*

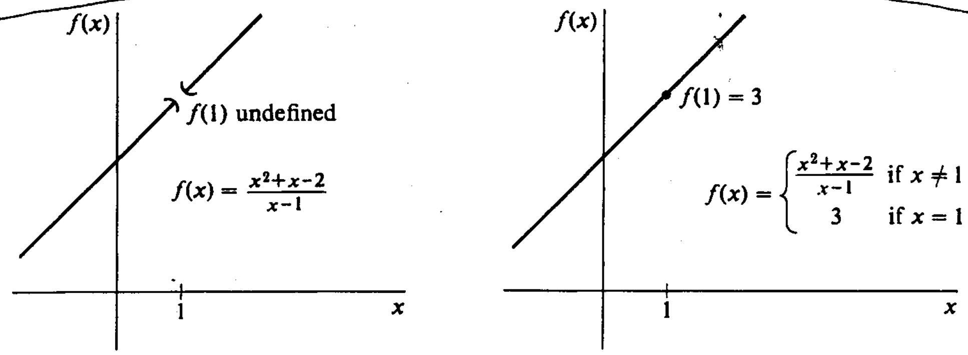


Figure 3.4.2

To summarize, given a function $y = f(x)$ defined at $x = c$, all the statements below are equivalent.

(1) f is continuous at c.
(2) Whenever $x \approx c$, $f(x) \approx f(c)$.
(3) Whenever $st(x) = c$, $st(f(x)) = f(c)$.
(4) $\lim_{x \to c} f(x) = f(c)$.
(5) y is continuous at $x = c$.
(6) Whenever Δx is infinitesimal, Δy is infinitesimal.

Our next theorem is that differentiability implies continuity. That is, the set of differentiable functions at c is a subset of the set of continuous functions at c. (See Figure 3.4.3.)

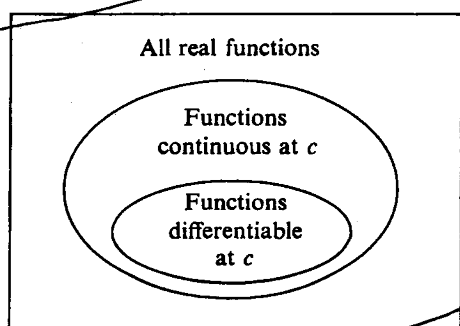


Figure 3.4.3

THEOREM 3

If f is differentiable at c then f is continuous at c.

PROOF Let $y = f(x)$, and let Δx be a nonzero infinitesimal. Then $\Delta y/\Delta x$ is infinitely close to $f'(c)$ and is therefore finite. Thus $\Delta y = \Delta x(\Delta y/\Delta x)$ is the product of an infinitesimal and a finite number, so Δy is infinitesimal.

For example, the transcendental functions $\sin x$, $\cos x$, e^x are continuous for all x, and $\ln x$ is continuous for $x > 0$. Theorem 3 can be used to show that combinations of these functions are continuous.

EXAMPLE 2 Find as large a set as you can on which the function

$$f(x) = \frac{\sin x \ln(x+1)}{x^2 - 4}$$

is continuous.

$\sin x$ is continuous for all x. $\ln(x+1)$ is continuous whenever $x + 1 > 0$, that is, $x > -1$. The numerator $\sin x \ln(x+1)$ is thus continuous whenever $x > -1$. The denominator $x^2 - 4$ is continuous for all x but is zero when $x = \pm 2$. Therefore $f(x)$ is continuous whenever $x > -1$ and $x \neq 2$.

The next two examples give functions which are continuous but *not* differentiable at a point c.

EXAMPLE 3 The function $y = x^{1/3}$ is continuous but not differentiable at $x = 0$. (See Figure 3.4.4(a).) We have seen before that it is not differentiable at $x = 0$. It is continuous because if Δx is infinitesimal then so is

$$\Delta y = (0 + \Delta x)^{1/3} - 0^{1/3} = (\Delta x)^{1/3}.$$

EXAMPLE 4 The absolute value function $y = |x|$ is continuous but not differentiable at the point $x = 0$. (See Figure 3.4.4(b).)

We have already shown that the derivative does not exist at $x = 0$. To see that the function is continuous, we note that for any infinitesimal Δx,

$$\Delta y = |0 + \Delta x| - |0| = |\Delta x|$$

and thus Δy is infinitesimal.

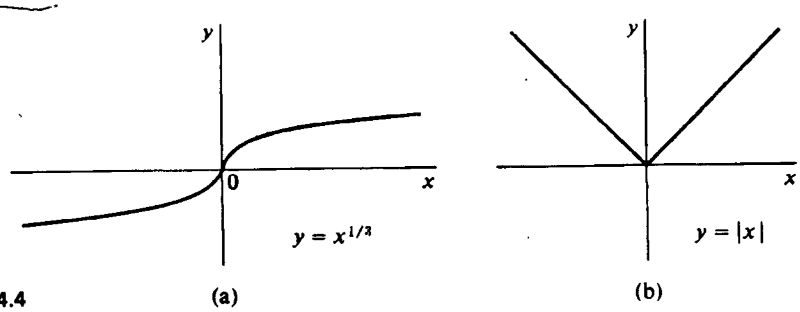


Figure 3.4.4

The path of a bouncing ball is a series of parabolas shown in Figure 3.4.5. The curve is continuous everywhere. At the points $a_1, a_2, a_3, \ldots$ where the ball bounces, the curve is continuous but not differentiable. At other points, the curve is both continuous and differentiable.

In the classical kinetic theory of gases, a gas molecule is assumed to be moving at a constant velocity in a straight line except at the instant of time when it collides with another molecule or the wall of the container. Its path is then a broken line in space, as in Figure 3.4.6.