(1)

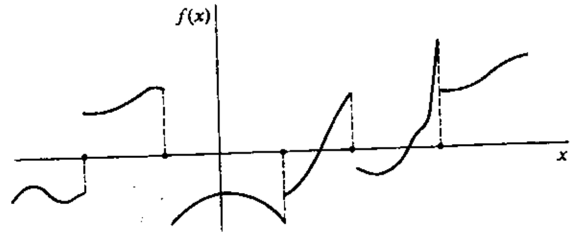


Figure 3.4.8 Points where f is discontinuous

The next theorem is similar to the Chain Rule for derivatives.

THEOREM 4

If f is continuous at c and G is continuous at $f(c)$, then the function

$$g(x) = G(f(x))$$

is also continuous at c. That is, a continuous function of a continuous function is continuous.

PROOF Let x be infinitely close to but not equal to c. Then

$$st(g(x)) = st(G(f(x))) = G(st(f(x))) = G(f(c)) = g(c).$$

For example, the following functions are continuous:

$$\begin{array}{ll} f(x) = \sqrt{x^2+1}, & \text{all } x \\ g(x) = |x^3 - x|, & \text{all } x \\ h(x) = (1+\sqrt{x})^{1/3}, & x > 0 \\ j(x) = e^{\sin x}, & \text{all } x \\ k(x) = \ln|x|, & \text{all } x \neq 0 \end{array}$$

Here are two examples illustrating two types of discontinuities.

ערך השלם

EXAMPLE 5 The function $g(x) = \dfrac{x^2 - 3x + 4}{4(x-1)(x-2)}$ is continuous at every real point except $x = 1$ and $x = 2$. At these two points $g(x)$ is undefined (Figure 3.4.9).

EXAMPLE 6 The *greatest integer function* $[x]$, shown in Figure 3.4.10, is defined by

$$[x] = \text{the greatest integer } n \text{ such that } n \leq x.$$

Thus $[x] = 0$ if $0 \leq x < 1$, $[x] = 1$ if $1 \leq x < 2$, $[x] = 2$ if $2 \leq x < 3$, and so on. For negative x, we have $[x] = -1$ if $-1 \leq x < 0$, $[x] = -2$

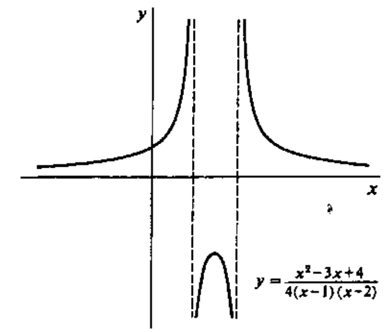


Figure 3.4.9

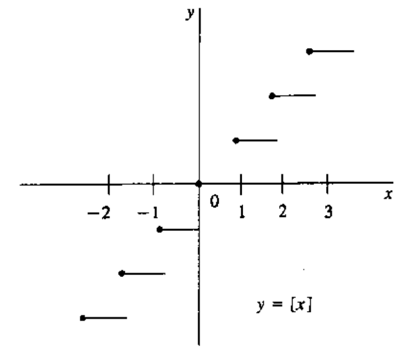


Figure 3.4.10

if $-2 \leq x < -1$, and so on. For example,

$$[7.362] = 7, \quad [\pi] = 3, \quad [-2.43] = -3.$$

For each integer n, $[n]$ is equal to n. The function $[x]$ is continuous when x is not an integer but is discontinuous when x is an integer n. At an integer n, both one-sided limits exist but are different,

$$\lim_{x \to n^-} f(x) = n - 1, \qquad \lim_{x \to n^+} f(x) = n.$$

The graph of $[x]$ looks like a staircase. It has a step, or jump discontinuity, at each integer n. The function $[x]$ will be useful in the last section of this chapter. Some hand calculators have a key for either the greatest integer function or for the similar function that gives $[x]$ for positive x and $[x] + 1$ for negative x.

↓ 26 dec

Functions which are "continuous on an interval" will play an important role in this chapter. Intervals were discussed in Section 1.1. Recall that closed intervals have the form

$$[a, b].$$

open intervals have one of the forms

$$(a, b), \quad (a, \infty), \quad (-\infty, b), \quad (-\infty, \infty),$$

and half-open intervals have one of the forms

$$[a, b), \quad (a, b], \quad [a, \infty), \quad (-\infty, b].$$

In these intervals, a is called the *lower endpoint* and b, the *upper endpoint*. The symbol $-\infty$ indicates that there is no lower endpoint, while ∞ indicates that there is no upper endpoint.

DEFINITION

*We say that f is **continuous on an open interval** I if f is continuous at every point c in I. If in addition f has a derivative at every point of I, we say that f is **differentiable on** I.*

To define what is meant by a function continuous on a closed interval, we introduce the notions of continuous from the right and continuous from the left, using one-sided limits.

DEFINITION

*f is **continuous from the right** at c if $\lim_{x \to c^+} f(x) = f(c)$.*

*f is **continuous from the left** at c if $\lim_{x \to c^-} f(x) = f(c)$.*

EXAMPLE 6 (Continued) The greatest integer function $f(x) = [x]$ is continuous from the right but not from the left at each integer n because

$$[n] = n, \qquad \lim_{x \to n^+}[x] = n, \qquad \lim_{y \to n^-}[x] = n - 1.$$

It is easy to check that f is continuous at c if and only if f is continuous from both the right and left at c.

DEFINITION

*f is said to be **continuous on the closed interval** $[a, b]$ if f is continuous at each point c where $a < c < b$, continuous from the right at a, and continuous from the left at b.*

Figure 3.4.11 shows a function f continuous on $[a, b]$.

EXAMPLE 7 The semicircle

$$y = \sqrt{1 - x^2},$$

shown in Figure 3.4.12, is continuous on the closed interval $[-1, 1]$. It is

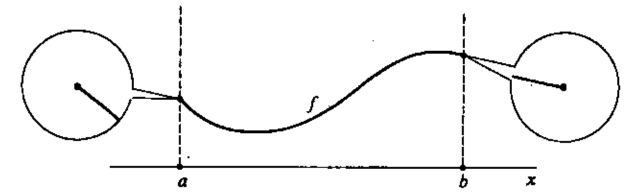


Figure 3.4.11 f is continuous on the interval $[a, b]$

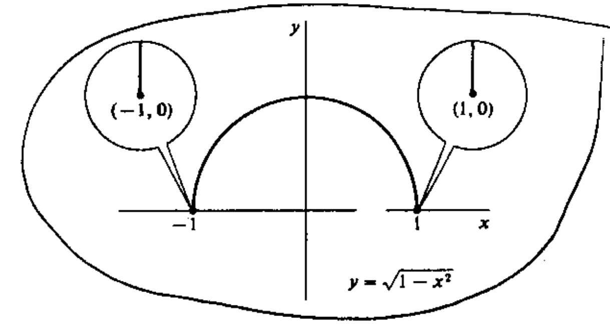


Figure 3.4.12

differentiable on the open interval $(-1, 1)$. To see that it is continuous from the right at $x = -1$, let Δx be positive infinitesimal. Then

$$y = \sqrt{1 - (-1)^2} = 0$$

$$y + \Delta y = \sqrt{1 - (-1 + \Delta x)^2} = \sqrt{1 - (1 - 2\Delta x + \Delta x^2)}$$

$$= \sqrt{2\Delta x - \Delta x^2} = \sqrt{(2 - \Delta x)\Delta x}.$$

Thus

$$\Delta y = \sqrt{(2 - \Delta x)\Delta x}.$$

The number inside the radical is positive infinitesimal, so Δy is infinitesimal. This shows that the function is continuous from the right at $x = -1$. Similar reasoning shows it is continuous from the left at $x = 1$.

31 dec

PROBLEMS FOR SECTION 3.4

In Problems 1–17, find the set of all points at which the function is continuous.

1 $f(x) = 3x^2 + 5x + 4$

2 $f(x) = \dfrac{5x + 2}{x^2 + 1}$

3 $f(x) = \sqrt{x + 2}$

4 $f(x) = \dfrac{x}{x + 2}$

5 $f(x) = \sqrt{|x - 2| + 1}$

6 $f(x) = \dfrac{x + 3}{|x + 3|}$

7 $f(x) = \dfrac{x}{x^2 + x}$

8 $f(x) = \dfrac{x + 2}{(x - 1)(x - 3)^{1/3}}$

9 $f(x) = \sqrt{4 - x^2}$

10 $f(x) = \sqrt{x^2 - 4}$

11 $f(x) = \dfrac{1}{x - (1/(x + 1))}$

12 $g(x) = \dfrac{1}{x} + \dfrac{1}{x - 1}$

13 $g(x) = \dfrac{x - 2}{x - 3} + \dfrac{x - 3}{x - 2}$

14 $g(x) = \sqrt{x^3 - x}$

15 $g(x) = \sqrt[4]{x^2 - x^3}$

16 $f(t) = \sqrt{t^{-2} - 1}$

17 $f(t) = \sqrt{t^{-1} - 1}$

18 Show that $f(x) = \sqrt{x}$ is continuous from the right at $x = 0$.

19 Show that $f(x) = \sqrt{1 - x}$ is continuous from the left at $x = 1$.

20 Show that $f(x) = \sqrt{1 - |x|}$ is continuous on the closed interval $[-1, 1]$.

21 Show that $f(x) = \sqrt{x} + \sqrt{2 - x}$ is continuous on the closed interval $[0, 2]$.

22 Show that $f(x) = \sqrt{9 - x^2}$ is continuous on the closed interval $[-3, 3]$.

23 Show that $f(x) = \sqrt{x^2 - 9}$ is continuous on the half-open intervals $(-\infty, -3]$ and $[3, \infty)$.

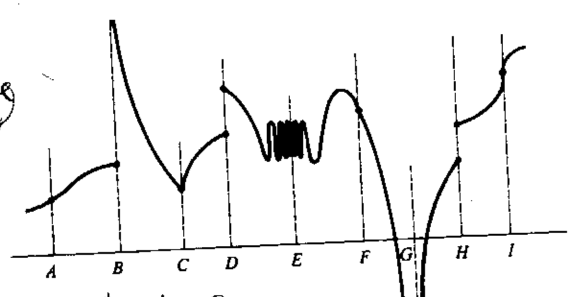
☐ 24 Suppose the function $f(x)$ is continuous on the closed interval $[a, b]$. Show that there is a function $g(x)$ which is continuous on the whole real line and has the value $g(x) = f(x)$ for x in $[a, b]$.

☐ 25 Suppose $\lim_{x \to c} f(x) = L$. Prove that the function $g(x)$, defined by $g(x) = f(x)$ for $x \neq c$ and $g(x) = L$ for $x = c$, is continuous at c.

26 In the curve $y = f(x)$ illustrated below, identify the points $x = c$ where each of the following happens:

(a) f is discontinuous at $x = c$
(b) f is continuous but not differentiable at $x = c$.

mnemonic device
[illegible]



Review definitions of continuity ① ② ③ ④

3.5 MAXIMA AND MINIMA

Let us assume throughout this section that f is a real function whose domain is an interval I, and furthermore that f is continuous on I. A problem that often arises is that of finding the point c where $f(c)$ has its largest value, and also the point c where $f(c)$ has its smallest value. The derivative turns out to be very useful in this problem. We begin by defining the concepts of maximum and minimum.

DEFINITION

Let c be a real number in the domain I of f.

(i) *f has a **maximum** at c if $f(c) \geq f(x)$ for all real numbers x in I. In this case $f(c)$ is called the **maximum value** of f.*

(ii) *f has a **minimum** at c if $f(c) \leq f(x)$ for all real numbers x in I. $f(c)$ is then called the **minimum value** of f.*

When we look at the graph of a continuous function f on I, the maximum will appear as the highest peak and the minimum as the lowest valley (Figure 3.5.1).

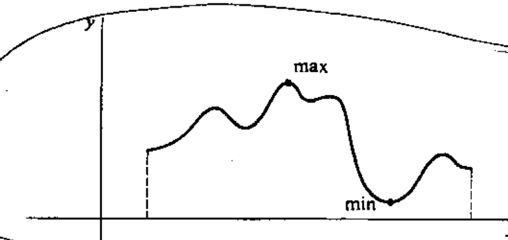


Figure 3.5.1 Maximum and Minimum

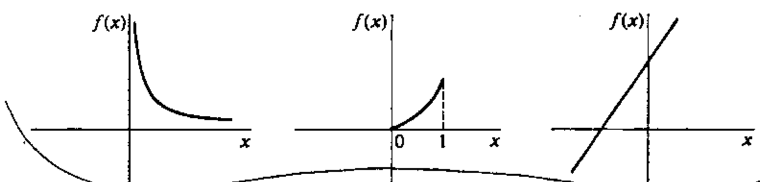
In general, all of the following possibilities can arise:

f has no maximum in its domain I.
f has a maximum at exactly one point in I.
f has a maximum at several different points in I.

However even if f has a maximum at several different points, f can have only one maximum value. Because if f has a maximum at c_1 and also at c_2, then $f(c_1) \geq f(c_2)$ and $f(c_2) \geq f(c_1)$, and therefore $f(c_1)$ and $f(c_2)$ are equal.

EXAMPLE 1 Each of the following functions, graphed in Figure 3.5.2, have no maximum and no minimum:

(a) $f(x) = 1/x, \quad 0 < x$.
(b) $f(x) = x^2, \quad 0 < x < 1$.
(c) $f(x) = 2x + 3$.



(a) $f(x) = \frac{1}{x}, 0 < x$ (b) $f(x) = x^2, 0 < x < 1$ (c) $f(x) = 2x + 3$

Figure 3.5.2 No Maximum or Minimum

EXAMPLE 2 The function $f(x) = x^2 + 1$ has no maximum. But f has a minimum at $x = 0$ with value 1, because for $x \neq 0$, we always have $x^2 > 0, x^2 + 1 > 1$. The graph is shown in Figure 3.5.3.

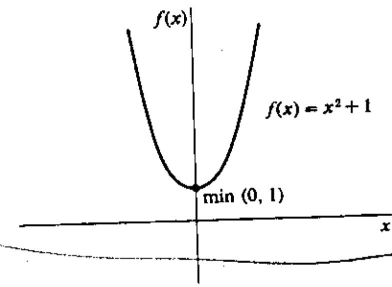


Figure 3.5.3

The use of the derivative in finding maxima and minima is based on the Critical Point Theorem. It shows that the maxima and minima of a function can only occur at certain points, called *critical points*. The theorem will be stated now, and its proof is given at the end of this section.

CRITICAL POINT THEOREM Also known as Fermat's theorem

Let f be continuous on its domain I. Suppose that c is a point in I and f has either a maximum or a minimum at c. Then one of the following three things must happen:

(i) *c is an endpoint of I,*
(ii) *$f'(c)$ is undefined,*
(iii) *$f'(c) = 0$.*

proof given after page 142

Def.

We shall say that c is a *critical point* of f if either (i), (ii), or (iii) happens. The three types of critical points are shown in Figure 3.5.4. When I is an open interval, (i) cannot arise since the endpoints are not elements of I. But when I is a closed interval, the two endpoints of I will always be among the critical points. Geometrically the theorem says that if f has a maximum or minimum at c, then either c is an endpoint of the curve, or there is a sharp corner at c, or the curve has a horizontal slope at c. Thus at a maximum there is either an endpoint, a sharp peak, or a horizontal summit.

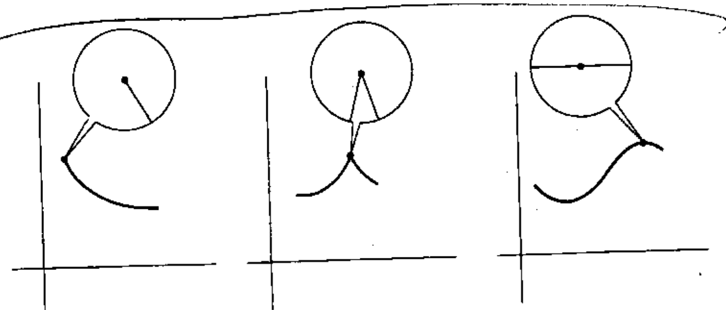


Figure 3.5.4 Critical Point Theorem

The Critical Point Theorem has some important applications to economics. Here is one example. Some other examples are described in the problem set.

EXAMPLE 3 Suppose a quantity x of a commodity can be produced at a total cost $C(x)$ and sold for a total revenue of $R(x)$, $0 < x < \infty$. The *profit* is defined as the difference between the revenue and the cost,

$$P(x) = R(x) - C(x).$$

Show that if the profit has a maximum at x_0, then the marginal cost is equal to the marginal revenue at x_0,

$$R'(x_0) = C'(x_0).$$

In this problem it is understood that $R(x)$ and $C(x)$ are differentiable functions, so that the marginal cost and marginal revenue always exist. Therefore $P'(x)$ exists and

$$P'(x) = R'(x) - C'(x).$$

Assume $P(x)$ has a maximum at x_0. Since $(0, \infty)$ has no endpoints and $P'(x_0)$ exists, the Critical Point Theorem shows that $P'(x_0) = 0$. Thus

$$P'(x_0) = R'(x_0) - C'(x_0) = 0$$

and

$$R'(x_0) = C'(x_0).$$

DEFINITION

*An **interior point** of an interval I is an element of I which is not an endpoint of I.*

For example, if I is an open interval, then every point of I is an interior point of I. But if I is a closed interval $[a, b]$, then the set of all interior points of I is the open interval (a, b) (Figure 3.5.5).

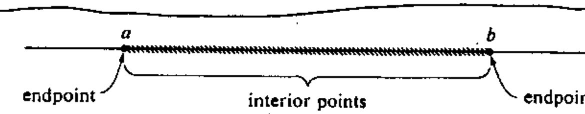


Figure 3.5.5

An interior point of I which is a critical point of f is called an *interior critical point*. There are a number of tests to determine whether or not f has a maximum at a given interior critical point. Here are two such tests. In both tests we assume that f is continuous on its domain I.

DIRECT TEST

Suppose c is the only interior critical point of f, and u, v are points in I with $u < c < v$.

(i) *If $f(c) > f(u)$ and $f(c) > f(v)$, then f has a maximum at c and nowhere else.*

(ii) *If $f(c) < f(u)$ and $f(c) < f(v)$, then f has a minimum at c and nowhere else.*

(iii) *Otherwise, f has neither a maximum nor a minimum at c.*

The three cases in the Direct Test are shown in Figure 3.5.6. The advantage of the Direct Test is that one can determine whether f has a maximum or minimum at c by computing only the three values $f(u)$, $f(v)$, and $f(c)$ instead of computing all values of $f(x)$.

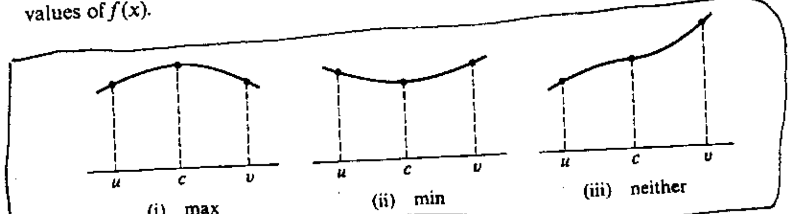


Figure 3.5.6

PROOF OF THE DIRECT TEST We must prove that if two points of I are on the same side of c, their values are on the same side of $f(c)$. Suppose, for instance, that $u_1 < u_2 < c$ (Figure 3.5.7). On the closed interval $[u_1, c]$ the only

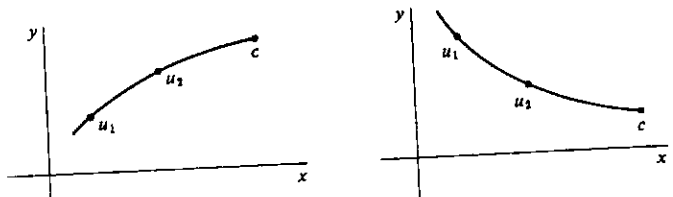


Figure 3.5.7

critical points are the endpoints. Thus when we restrict f to this interval, it has a maximum at one endpoint and a minimum at the other. If the maximum is at c, then $f(u_1)$ and $f(u_2)$ are both less than $f(c)$; if the minimum is at c, then $f(u_1)$ and $f(u_2)$ are both greater than $f(c)$. A similar proof works when $c < v_1 < v_2$.

SECOND DERIVATIVE TEST

Suppose c is the only interior critical point of f and that $f'(c) = 0$.

(i) *If $f''(c) < 0$, f has a maximum at c and nowhere else.*

(ii) *If $f''(c) > 0$, f has a minimum at c and nowhere else.*

We omit the proof and give a simple intuitive argument instead. (See Figure 3.5.8.) Since $f'(c) = 0$, the curve is horizontal at c. If $f''(c)$ is negative the slope is decreasing. This means that the curve climbs up until it levels off at c and then falls down, so it has a maximum at c. On the other hand, if $f''(c)$ is positive, the slope is increasing, so the curve falls down until it reaches a minimum at c and then climbs up. This argument makes it easy to remember which way the inequalities go in the test.

The Second Derivative Test fails when $f''(c) = 0$ and when $f''(c)$ does not exist. When the Second Derivative Test fails any of the following things can still happen:

(1) f has a maximum at $x = c$.
(2) f has a minimum at $x = c$.
(3) f has neither a maximum nor a minimum at $x = c$.

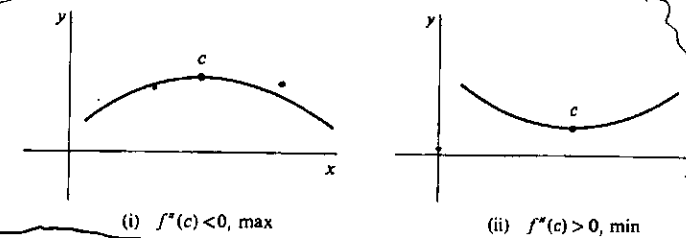


Figure 3.5.8

In most maximum and minimum problems, there is only one critical point except for the endpoints of the interval. We develop a method for finding the maximum and minimum in that case.

METHOD FOR FINDING MAXIMA AND MINIMA

When to use: f is continuous on its domain I, and f has exactly one interior critical point.

Step 1 Differentiate f.

Step 2 Find the unique interior critical point c of f.

Step 3 Test to see whether f has a maximum or minimum at c. The Direct Test or the Second Derivative Test may be used.

This method can be applied to an open or half-open interval as well as a closed interval. The Second Derivative Test is more convenient because it requires only the single computation $f''(c)$, while the Direct Test requires the three computations $f(u)$, $f(v)$, and $f(c)$. However, the Direct Test always works while the Second Derivative Test sometimes fails.

We illustrate the use of both tests in the examples.

EXAMPLE 4 Find the point on the line $y = 2x + 3$ which is at minimum distance from the origin.

The distance is given by

$$z = \sqrt{x^2 + y^2},$$

and substituting $2x + 3$ for y,

$$z = \sqrt{x^2 + (2x+3)^2} = \sqrt{5x^2 + 12x + 9}.$$

This is defined on the whole real line.

Step 1 $\boxed{\dfrac{dz}{dx} = \dfrac{10x + 12}{2\sqrt{5x^2 + 12x + 9}} = \dfrac{5x+6}{z}.}$ ← Convenient form

Step 2 $\dfrac{dz}{dx} = 0$ only when $5x + 6 = 0$, or $x = -\frac{6}{5}$.

Step 3 $\dfrac{d^2z}{dx^2} = \dfrac{5z - (5x+6)(dz/dx)}{z^2}$.

At $x = -\frac{6}{5}$, $5x + 6 = 0$ and $z > 0$ so $d^2z/dx^2 = 5/z > 0$. By the Second Derivative Test, z has a minimum at $x = -\frac{6}{5}$.

CONCLUSION The distance is a minimum at $x = -\frac{6}{5}$, $y = 2x + 3 = \frac{3}{5}$. The minimum distance is $z = \sqrt{x^2 + y^2} = \sqrt{\frac{9}{5}}$. This is shown in Figure 3.5.9.

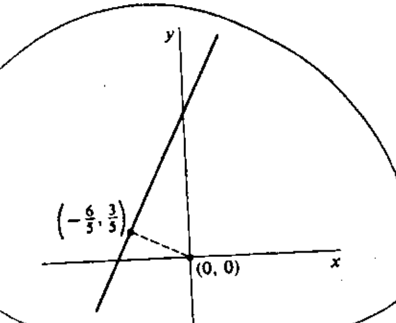


Figure 3.5.9

1 to here 2 jdn

EXAMPLE 5 Find the minimum of $f(x) = x^6 + 10x^4 + 2$.

Step 1 $f'(x) = 6x^5 + 40x^3 = x^3(6x^2 + 40)$.

Step 2 $f'(x) = 0$ only when ~~$x = 0$~~.

Step 3 The Second Derivative Test fails, because

$$f''(x) = 30x^4 + 120x^2, \qquad f''(0) = 0.$$

We use the Direct Test. Let $u = -1, v = 1$. Then

$$f(0) = 2, \qquad f(-1) = 13, \qquad f(1) = 13.$$

Hence ~~f has a minimum at 0~~, as shown in Figure 3.5.10.

EXAMPLE 6 Find the maximum of $f(x) = 1 - x^{2/3}$.

Step 1 $f'(x) = -(\frac{2}{3})x^{-1/3}$.

Step 2 $f'(x)$ is undefined at $x = 0$, and this the only critical point.

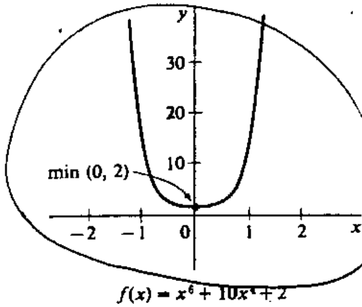


Figure 3.5.10

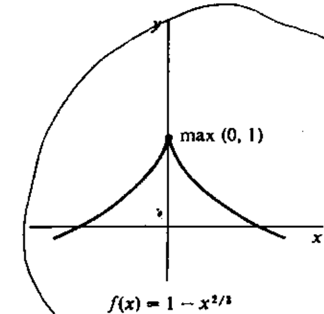


Figure 3.5.11

Step 3 We use the Direct Test. Let $u = -1, v = 1$.

$$f(0) = 1, \qquad f(-1) = 0, \qquad f(1) = 0.$$

Thus f has a maximum at $x = 0$, as shown in Figure 3.5.11.

If f has more than one interior critical point, the maxima and minima can sometimes be found by dividing the interval into two or more parts.

EXAMPLE 7 Find the maximum and minimum of $f(x) = x/(x^2 + 1)$.

Step 1 $f'(x) = \dfrac{(x^2+1) - 2x^2}{(x^2+1)^2} = \dfrac{1 - x^2}{(x^2+1)^2}$.

Step 2 $f'(x) = 0$, when $x = -1$ and $x = 1$. There are two interior critical points. We divide the interval $(-\infty, \infty)$ on which f is defined into the two subintervals $(-\infty, 0]$ and $[0, \infty)$. On each of these subintervals, f has just one interior critical point.

Step 3 We shall use the direct test for the subinterval $(-\infty, 0]$. At the critical point -1, we have $f(-1) = -\frac{1}{2}$. By direct computation, we see that $f(-2) = -\frac{2}{5}$ and $f(0) = 0$. Both of these values are greater than $-\frac{1}{2}$. This shows that the restriction of f to the subinterval $(-\infty, 0]$ has a minimum at $x = -1$. Moreover, $f(x)$ is always ≥ 0 for x in the other subinterval $[0, \infty)$. Therefore f has a minimum at -1 for the whole interval $(-\infty, \infty)$. In a similar way, we can show that f has a maximum at $x = 1$.

CONCLUSION f has a minimum at $x = -1$ with value $f(-1) = -\frac{1}{2}$, and a maximum at $x = 1$ with value $f(1) = \frac{1}{2}$. (See Figure 3.5.12.)

The Critical Point Theorem can often be used to show that a curve has no maximum or minimum on an open interval $I = (a, b)$. The theorem shows that:

If $y = f(x)$ has no critical points in (a, b), the curve has no maximum or minimum on (a, b).

If $y = f(x)$ has just one critical point $x = c$ in (a, b) and two points x_1 and

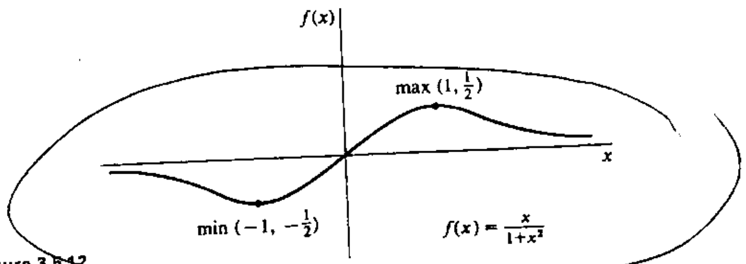


Figure 3.5.12

x_2 *are found where* $f(x_1) < f(c) < f(x_2)$, *then the curve has no maximum or minimum on* (a, b).

EXAMPLE 8 $f(x) = x^3 - 1$. Test for maxima and minima.

Step 1 $f'(x) = 3x^2$.

Step 2 $f'(x) = 0$ only when $x = 0$.

Step 3 The Second Derivative Test fails because $f''(x) = 6x$, $f''(0) = 0$.

By direct computation, $f(0) = -1$, $f(-1) = -2$, $f(1) = 0$.

Therefore f has neither a minimum nor a maximum at $x = 0$.

CONCLUSION Since $x = 0$ is the only critical point of f and f doesn't have a maximum or minimum there, we conclude that f has no maximum and no minimum as shown in Figure 3.5.13.

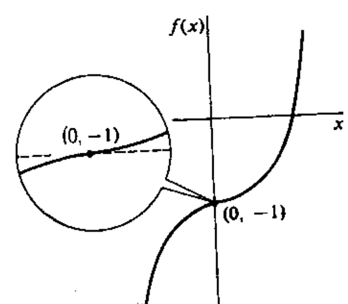


Figure 3.5.13

PROOF OF THE CRITICAL POINT THEOREM Assume that neither (i) nor (ii) holds; that is, assume that c is not an endpoint of I and $f'(c)$ exists. We must show that (iii) is true; i.e., $f'(c) = 0$. We give the proof for the case that f has a maximum at c. Let $x = c$, and let $\Delta x > 0$ be infinitesimal. Then

$$f(c + \Delta x) \le f(c), \qquad f(c - \Delta x) \le f(c).$$

(See Figure 3.5.14.) Therefore

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} \le 0 \le \frac{f(c - \Delta x) - f(c)}{-\Delta x}.$$

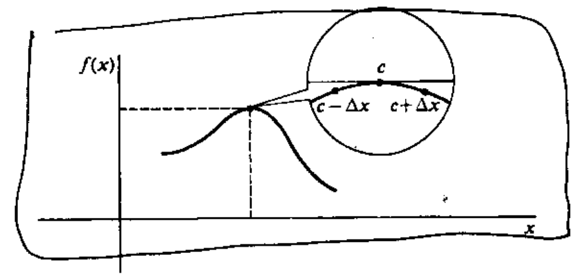


Figure 3.5.14 Proof of the Critical Point Theorem

Taking standard parts,

$$f'(c) = st\left(\frac{f(c + \Delta x) - f(c)}{\Delta x}\right) \le 0,$$

and also,

$$0 \le st\left(\frac{f(c - \Delta x) - f(c)}{-\Delta x}\right) = f'(c).$$

Therefore $f'(c) = 0$.

PROBLEMS FOR SECTION 3.5

In Problems 1–36, find the unique interior critical point and determine whether it is a maximum, a minimum, or neither.

1	$f(x) = x^2$	2	$f(x) = 1 - x^2$
3	$f(x) = x^4 + 2$	4	$f(x) = x^4 + 3x^2 + 5$
5	$f(x) = x^3 + 2$	6	$f(x) = x^3 - 3x^2 + 3x$
7	$f(x) = 3x^2 + 2x - 5$	8	$f(x) = 2(x - 1)^4 + (x - 1)^2 + 6$
9	$f(x) = x^{4/5}$	10	$f(x) = 2 - (x + 1)^{2/3}$
11	$f(x) = \dfrac{1}{x^2 - 1}$, $-1 < x < 1$	12	$f(x) = \dfrac{1}{x^2 + 1}$
13	$f(x) = x^{1/3} + 1$	14	$f(x) = 4 - x^{1/5}$
15	$f(x) = x^2 - x^{-1}$, $x < 0$	16	$f(x) = x^2 - x^{-1}$, $x > 0$
17	$f(x) = x^{-1} - (x - 3)^{-1}$, $0 < x < 3$	18	$f(x) = x + x^{-1}$, $0 < x$
19	$f(x) = \sqrt{4 - x^2}$, $-2 \le x \le 2$	20	$f(x) = (4 - x^2)^{-1/2}$, $-2 < x < 2$
21	$y = \sin x + x$, $0 \le x \le 2\pi$	22	$y = \sin^2 x$, $0 < x < \pi$
23	$y = e^{-x^2}$	24	$y = e^{x^2 - 1}$
25	$y = \dfrac{1}{\cos x}$, $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$	26	$y = \ln(\sin x)$, $0 < x < \pi$
27	$y = xe^x$	28	$y = x\ln x$, $0 < x < \infty$
29	$y = x - \ln x$, $0 < x < \infty$	30	$y = e^x - x$
31	$f(x) = \lvert x - 3\rvert$	32	$f(x) = 3 + \lvert 1 - x\rvert$
33	$f(x) = 2 - \lvert x\rvert$	34	$f(x) = 2\lvert x\rvert - x$

35 $f(x) = \sqrt{x} + \sqrt{1-x}$, $0 \le x \le 1$

36 $f(x) = \sqrt{x} + \sqrt{9-3x}$, $0 \le x \le 3$

37 Find the shortest distance between the line $y = 1 - 4x$ and the origin.

38 Find the shortest distance between the curve $y = 2/x$ and the origin.

39 Find the minimum of the curve $f(x) = x^m - mx$, $x > 0$, where m is an integer ≥ 2.

40 Find the maximum of $f(x) = x^m - mx$, $x < 0$, where m is an odd integer ≥ 2.

In Problems 41–44, find the maximum and minimum of the given curve.

41 $f(x) = \dfrac{x}{x^2+4}$

42 $f(x) = \dfrac{3x+4}{x^2+1}$

43 $f(x) = \dfrac{x}{x^4+1}$

44 $f(x) = \dfrac{x^3}{x^4+1}$

3.6 MAXIMA AND MINIMA—APPLICATIONS

Maximum and minimum problems arise in both the physical and social sciences. We give three examples.

EXAMPLE 1 A woman wishes to rent a house. If she lives x miles from her work, her transportation cost will be cx dollars per year, while her rent will be $25c/(x+1)$ dollars per year. How far should she live from work to minimize her rent and transportation expenses?

Let y be her expenses in dollars per year. Then

$$y = cx + \frac{25c}{x+1}.$$

The problem is to find the minimum value of y in the interval $0 \le x < \infty$.

Step 1 $\dfrac{dy}{dx} = c - \dfrac{25c}{(x+1)^2}$.

Step 2 To find x such that $dy/dx = 0$ we set $dy/dx = 0$ and solve for x.

$$c - \frac{25c}{(x+1)^2} = 0, \quad c = \frac{25c}{(x+1)^2}, \quad (x+1)^2 = 25, \quad x+1 = \pm 5.$$

Then $x = 4$ or $x = -6$. We reject $x = -6$ because $0 \le x$. The only interior critical point is $x = 4$.

Step 3 We use the Direct Test.

At $x = 0$, $y = c \cdot 0 + 25c/(0+1) = 25c$.

At $x = 4$, $y = 4c + 25c/(4+1) = 9c$.

At $x = 9$, $y = 9c + 25c/(9+1) = 11.5c$.

CONCLUSION y has its minimum at $x = 4$ miles. So the woman should live four miles from work. (See Figure 3.6.1.)

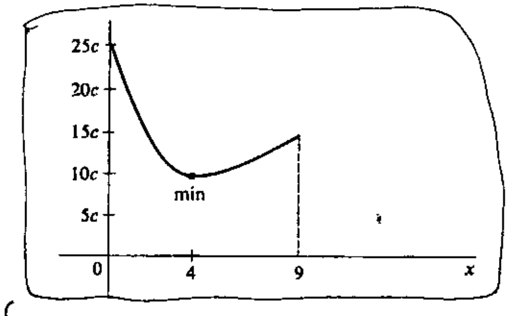


Figure 3.6.1

EXAMPLE 2 A farmer plans to use 1000 feet of fence to enclose a rectangular plot along the bank of a straight river. Find the dimensions which enclose the maximum area.

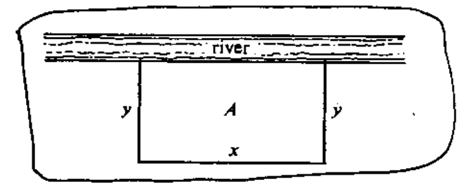


Figure 3.6.2

Let x be the dimension of the side along the river, and y be the other dimension, as in Figure 3.6.2. Call the area A.

No fencing is needed on the side of the plot bordering the river. The given information is expressed by the following system of formulas.

$$A = xy, \quad x + 2y = 1000, \quad 0 \le x \le 1000.$$

The problem is to find the values of x and y at which A is maximum. In this problem A is expressed in terms of two variables instead of one. However, we can select x as the independent variable, and then both y and A are functions of x. We find an equation for A as a function of x alone by eliminating y.

$$x + 2y = 1000, \qquad y = \frac{1000 - x}{2}.$$

$$A = xy = \frac{x(1000-x)}{2} = 500x - \frac{1}{2}x^2.$$

We then find the maximum of A in the closed interval $0 \le x \le 1000$.

Step 1 $dA/dx = 500 - x$.

Step 2 $dA/dx = 0$ when $x = 500$. This is the unique interior critical point.

Step 3 We use the Second Derivative Test: $d^2A/dx^2 = -1$. Therefore A has a maximum at the critical point $x = 500$.

CONCLUSION The maximum area occurs when the plot has dimensions $x = 500$ ft and $y = (1000 - x)/2 = 250$ ft (Figure 3.6.3).