$3\frac{1}{2}$

89-132 29 oct

חסם עליון — Least upper bound — chasam elyon

חסם תחתון — greatest lower bound — chasam tachton

מקסימום ומינימום — maximum and minimum

Example 1 Consider Set $S_1 = \{3 + \frac{4}{5n} \mid n \in \mathbb{N}\}$

The sequence $\frac{4}{5n}$ is decreasing. Therefore its maximal value is achieved for $n=1$: $\frac{4}{5n} = \frac{4}{5}$.

Similarly the maximal value of $3+\frac{4}{5n}$ is achieved for $n=1$, and equals $3+\frac{4}{5} = \frac{19}{5}$. Thus

$$\max(S) = \frac{19}{5}.$$

Meanwhile, the sequence $\frac{4}{5n}$ is decreasing and gets arbitrarily close to 0 as n tends to infinity. Therefore 0 is the chasam tachton of $\frac{4}{5n}$. And 3 is the chasam tachton of $3+\frac{4}{5n}$, so that

$$(\text{l.u.p.}) \ \inf(S) = 3.$$

Meanwhile we have strict inequality (i-shivyon mamash)

$$\frac{4}{5n} > 0$$

Hence

$$3 + \frac{4}{5n} > 3$$

Thus the infimum (l.u.p) is <u>not</u> attained.

In other words, the is <u>no minimum</u>.

Example 2 $S_2 = \{(-1)^n (2 - \frac{1}{n}) \mid n \in \mathbb{N}\}$.

Note that $2 - \frac{1}{n} > 0$ for all n. Therefore the element $u_n = (-1)^n (2 - \frac{1}{n})$ satisfies

$u_n < 0$ if n is odd,

$u_n > 0$ if n is even.

To determine the sup, we consider positive elements, namely those with n even. Thus, let $n = 2k$, and consider

$u_n = u_{2k} = (-1)^{2k}(2 - \frac{1}{2k}) = 2 - \frac{1}{2k}$. Here $2 - \frac{1}{2k}$ is decreasing

$\longrightarrow$

Therefore the maximal value is attained when $k=1$, i.e. $n=2$:

$u_2 = 2 - \frac{1}{2} = \frac{3}{2}$. Thus

$$\max(S_2) = \frac{3}{2}$$

and the supremum is attained: $\sup(S_2) = \max(S_2) = \frac{3}{2}$.

To determine the infimum of S_2, we will consider the negative terms u_n with n **odd**, i.e. $n = 2k-1$, $k \in \mathbb{N}$.

Then $u_n = u_{2k-1} = (-1)^{2k-1}\left(2 - \frac{1}{2k-1}\right) = -2 + \frac{1}{2k-1}$.

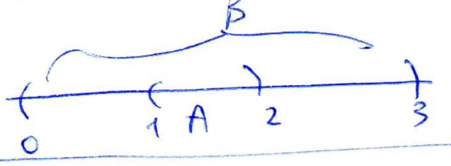
Here $\frac{1}{2k-1}$ is <u>decreasing</u> with k, and approaching 0. Then u_{2k-1} is approaching -2.

Therefore $\inf(S_2) = -2$. Meanwhile $\frac{1}{2k-1} > 0$

So $u_{2k-1} > -2$, therefore infimum is not attained.

<u>Example 3</u> Given Sets $A \subseteq B$, one has $\underbrace{\inf A}_{1} \geq \underbrace{\inf B}_{0}$; $\underbrace{\sup A}_{2} \leq \underbrace{\sup B}_{3}$

$A = [1,2]$, $B = [0,3)$.



<u>Ex 4</u> Now let $C = A \cap B$

Then $C \subseteq A$ and $C \subseteq B$.

Thus $\inf C \geq \inf A$, and also $\inf C \geq \inf B$.

Therefore $\inf C \geq \max(\inf A, \inf B)$.

<u>Example 5</u> Let $A = [-2,-1]$ and $B = [-4,-3]$.

Then define $C = \{ab \mid a \in A,\ b \in B\}$.

Then since $a < 0$ and $b < 0$, we have $ab > 0$.

Thus $\inf C = \inf(ab) = \inf\{|ab|\} = \inf\{|a|\,|b| \mid a \in A, b \in B\}$

Here $a \in [1,2]$ and $b \in [3,4]$. Hence $\inf C = 1 \cdot 3 = 3$.

$4\frac{1}{2}$

7 nov Sequences page 492 Keisler

Definition A sequence is a real function whose domain is the set of all positive integers.

A sequence a can be displayed as

$$a(1),\ a(2),\ a(3),\ \ldots,\ a(n),\ \ldots$$

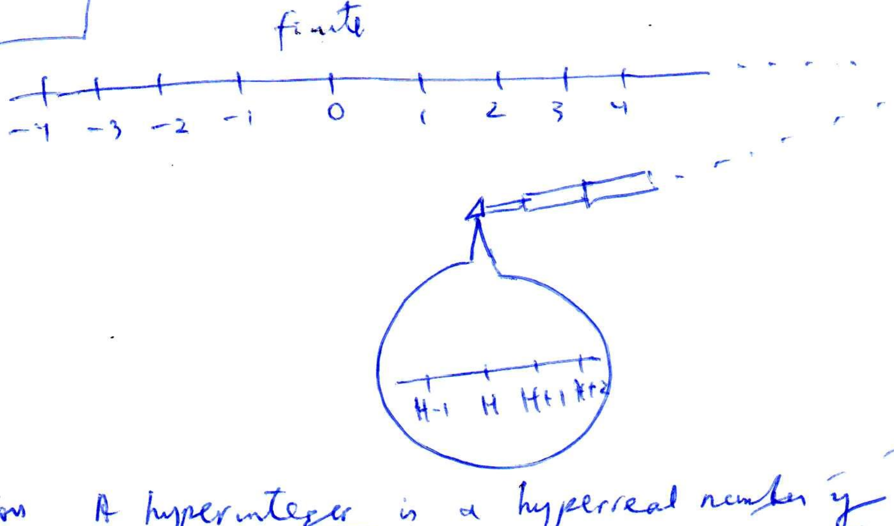
The value $a(n)$ is called the n-th term and written a_n.
The whole sequence is denoted following Keisler p. 492

$$\langle a_n \rangle = a_1, a_2, a_3, \ldots a_n, \ldots$$

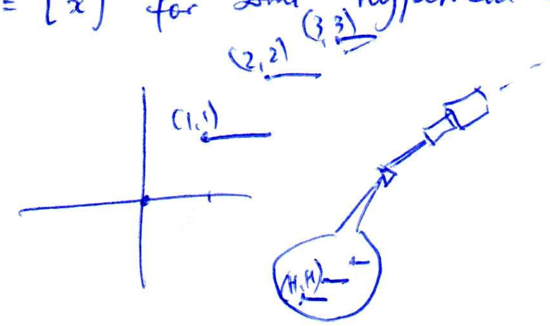
NB The metaselect uses notation $\{a_n\}$ but this is mostly used in Russia.

Remark Since a_n is defined for every positive integer n, a_H is defined for all positive infinite hyperinteger H. by the extension principle.

$\boxed{\text{back to section 3.8 page 159}}$



Definition A hyperinteger is a hyperreal number y such that $y = [x]$ for some hyperreal x.



By transfer principle, every hyperreal x is between two hyperintegers $[x] \le x < [x]+1$

Forward to page 493:

Examples of sequences

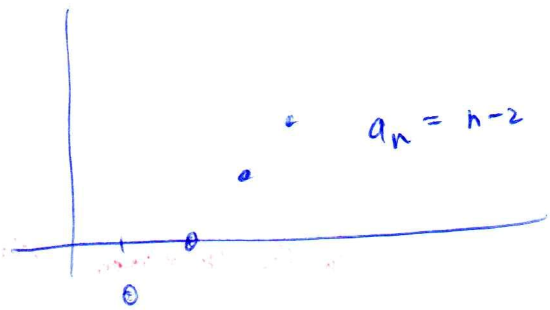
$1, 1, 1, 1, \ldots$ $\qquad a_n = 1$

$-1, 0, 1, 2, 3, \ldots$ $\qquad a_n = n-2$

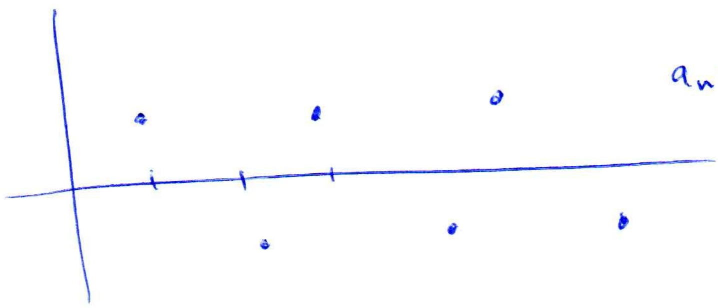
$-2, -4, -6, -8, -10, \ldots$ $\qquad a_n = -2n$

$1, -1, 1, -1, 1, \ldots$ $\qquad a_n = (-1)^{n-1}$

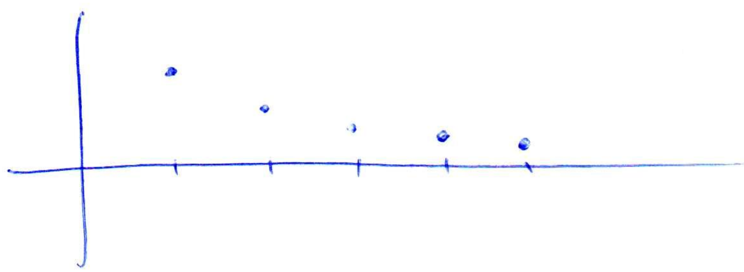
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ $\qquad a_n = \frac{1}{n}$



$a_n = n-2$



$a_n = (-1)^{n-1}$



$a_n = \frac{1}{n}$

Ex sequence

$$3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159, \ldots$$

is defined by the rule

$$a_n = \pi \text{ to } n \text{ decimal places}$$

that is, $a_n = \dfrac{m}{10^n}$ where m is integer s.t. $\dfrac{m}{10^n} \le \pi < \dfrac{m+1}{10^n}$

Ex. $n!$ factorial defined by

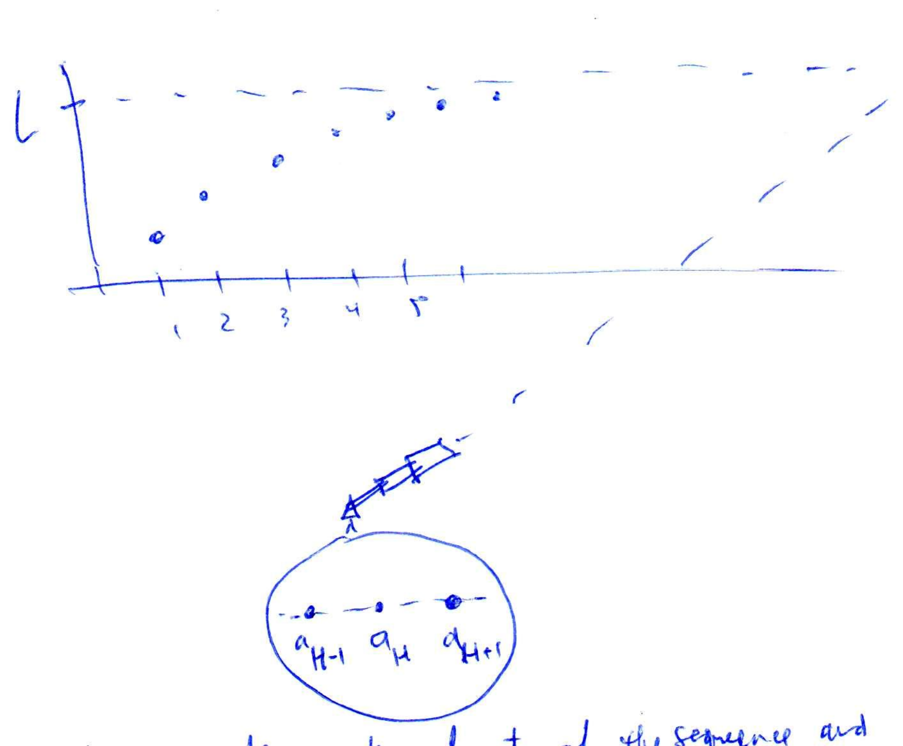
$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$$

The sequence $\langle n! \rangle$ is

$$1, 2, 6, 24, 120, 720, \ldots$$

By convention, $0! = 1$.

Definition A sequence $\langle a_n \rangle$ is said to converge to a real number L if a_H is infinitely close to L for all positive infinite hyperintegers H as in figure:



L is called the limit of the sequence and written $L = \lim_{n\to\infty} a_n$

Definition A sequence which does not converge to any real number is said to diverge (מתבדר).

Definition If a_H is positive infinite for all positive infinite H, the sequence is said to diverge to ∞ and we write

$$\lim_{n\to\infty} a_n = \infty.$$

NB. Sequences can diverge to $-\infty$, and also diverge without diverging to either ∞ or $-\infty$.

Criterion If we can find a_H and a_K which are not infinitely close then the sequence diverges.

Example 1 $\lim_{n\to\infty} 1 = 1$ converges because $a_H = 1$ for all H

Example 2 $\lim_{n\to\infty} n-2 = \infty$, diverges, because $H-2$ is positive infinite for all H.

Ex 3. $\lim_{n\to\infty} (-2n) = -\infty$. diverges, because $-2H$ is neg. inf

Ex 4 $\lim_{n\to\infty} (-1)^n$ is undefined. diverges because $(-1)^{2H} = 1$, $(-1)^{2H+1} = -1$

Ex 5 $\lim_{n\to\infty} \frac{1}{n} = 0$, converges, because $\frac{1}{H}$ has standard part 0.

Ex 6 Sequence $3.1,\ 3.14,\ 3.141,\ 3.1415,\ \ldots,\ a_n$, where $a_n =$ (π to n decimal places), converges to π. Then

$$\lim_{n\to\infty} a_n = \pi.$$

Proof. Let H be positive infinite. For some K,

$$\frac{K}{10^H} \le \pi < \frac{K+1}{10^H}.$$

Then $a_H = \frac{K}{10^H}$ and $a_H \le \pi < a_H + \frac{1}{10^H}$

Here $\frac{1}{10^H}$ is infinitesimal hence $a_H \approx \pi$ q.e.d.

Ex 7. $\lim_{n\to\infty} n! = \infty$

Ex 8. $\lim_{n\to\infty} \frac{4n^2+1}{n^2+3n} = st\left(\frac{4H^2+1}{H^2+3H}\right) = st\left(\frac{4+\frac{1}{H^2}}{1+\frac{3}{H}}\right) =$

$$= \frac{st\left(4+\frac{1}{H^2}\right)}{st\left(1+\frac{3}{H}\right)} = \frac{4+0}{1+0} = 4.$$

Ex 8. Let $c > 0$ be fixed. Find

$$\lim_{n\to\infty}\left(1+\frac{1}{c}\right)^n$$

p. 499 Theorem (ε, N condition for limits) We have

$$\lim_{n\to\infty} a_n = L$$

if and only if for every real $\varepsilon > 0$ $\exists$ positive integer N s.t. the numbers

$$a_N,\ a_{N+1},\ a_{N+2},\ \ldots\ a_{N+m},\ \ldots$$

are all within ε of L.

Example 9 $\lim \frac{2n^2}{3n+4} = \infty$

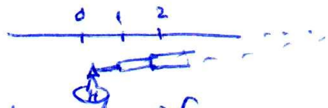
Ex 10 $\lim \left(\frac{5n}{6n^2+7}\right) = 0$

Ex 11 $\lim \frac{8-9n^2}{10n+11} = -\infty$

$5\frac{1}{2}$

Keisler p. 500 19 nov

Cauchy convergence test for sequences



Recall $\mathbb{N} \subseteq {}^*\mathbb{N}$ and telescope

A sequence $\langle a_n \rangle$ converges if and only if

$$a_H \approx a_K \quad \text{for all infinite } H \text{ and } K. \quad (*)$$

Definition A sequence $\langle a_n \rangle$ is Cauchy if it satisfies $(*)$.

Proof of test First suppose $\langle a_n \rangle$ converges, say to L: $\lim_{n\to\infty} a_n = L$.

Then for any infinite H and K,

$$a_H \approx L$$

as well as $a_K \approx L$

Therefore $a_H \approx a_K$ and the sequence is Cauchy.

Now suppose $\langle a_n \rangle$ is Cauchy. i.e. $(*)$ satisfied.

Case 1. assume a_H finite. Then for all infinite K,

$$st(a_K) = st(a_H)$$

So the sequence converges to the real number $st(a_H)$. qed.

Case 2. Suppose a_H is positive infinite. Then for each finite m, we have $a_H \geq a_m + 1$. Now consider the hyperintegers

$$\{1, 2, 3, \ldots H-1\}$$

Choose a largest element M such that $a_H \geq a_M + 1$.

This M cannot be infinite since $a_M \not\approx a_H$, and also cannot be finite since it must be bigger than all natural number. Contradiction.

Hence Case 2 cannot arise.

Case 3 Suppose a_H is negative infinite. By similar argument cannot arise

Thus only Case 1 is possible, and therefore the sequence converges.

Recall we always have an A-definition and a B-definition.

<u>A-definition of Cauchy sequence</u>: $\langle a_n \rangle$ is Cauchy if for all $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that if $m, n > m_\varepsilon$ then $|a_n - a_m| < \varepsilon$

<u>Recursive definition of a sequence</u> Given explicit function f.

either $a_{n+1} = f(a_n)$

or more generally $a_{n+1} = f(a_{n-1}, a_n)$ where $f = f(x,y)$ of 2 var, etc.

<u>Example</u>. Fibonacci sequence $F_0 = 0$, $F_1 = 1$, and

recursive relation $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.

Then get $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$

<u>Example</u>. $a_0 = 2$, $a_{n+1} = \sqrt{a_n}$. Here we can easily write down the general formula $a_1 = \sqrt{2}$, $a_2 = \sqrt[4]{2}$, $a_3 = \sqrt[8]{2}$, ..

$a_n = (2)^{\frac{1}{2^n}}$, and therefore $\lim_{n \to \infty} a_n = 1$.

In general, if the seq is defined by $a_{n+1} = f(a_n)$ and $\lim_{n \to \infty} a_n = L$ we look for L of the form $L = f(L)$.

In the case above, $L = \sqrt{L}$, hence $L^2 = L$,

$L^2 - L = 0$, $L(L-1) = 0$, so either $L = 0$ or $L = 1$.

Since all the term $a_n \geq 1$, it follows $L \geq 1$ hence $L = 1$ is the correct limit.

<u>Example</u> $a_0 = \sqrt{2}$, $a_{n+1} = \sqrt{2 a_n}$ To guess limit L, write $L = \sqrt{2L}$ hence $L^2 = 2L$, $L^2 - 2L = 0$, $L(L-2) = 0$. So either $L = 0$ or $L = 2$. But all $a_n \geq 1$

Hence must have $L = 2$ if the sequence converges.

Once we have guessed the limit, show that sequence is monotone increasing:

$1 \le a_0 \le 2$. By induction assume $1 \le a_n \le 2$ and show the same for a_{n+1}. But

$a_{n+1} = \sqrt{2a_n} \ge \sqrt{2} \ge 1$, and similarly

$a_{n+1} = \sqrt{2a_n} \le \sqrt{2 \cdot 2} = 2$. Thus $\forall_n\ 1 \le a_n \le 2$.

Now show sequence is increasing:

$$a_{n+1} - a_n = \sqrt{2a_n} - a_n = \sqrt{a_n}\left(\sqrt{2} - \sqrt{a_n}\right) \ge \sqrt{a_n}\left(\sqrt{2} - \sqrt{2}\right) = 0$$

Hence sequence is increasing and bounded.

Axiom of the real numbers: every bounded increasing sequence converges.

Proving that a sequence tends to 0 using upper bounds (δ's or pos)

Example. Find $\lim_{n \to \infty} \frac{1 \cos 1 + 4 \cos 2 + 9 \cos 3 + \dots + n^2 \cos n}{n^4}$

Use the fact that $|\cos x| \le 1$ for all x. Therefore letting $a_n = \frac{1 \cdot \cos 1 + \dots n^2 \cos n}{n^4}$, we see that

$$|a_n| \le \frac{|1 \cdot \cos 1| + 4 |\cos 2| + 9 |\cos 3| + \dots n^2 |\cos n|}{n^4}$$

Hence $|a_n| \le \frac{1 + 4 + 9 + \dots + n^2}{n^4}$

Now by problem set 1, exercise (7)(2) one has $1^2 + \dots + n^2 \le \frac{(n+1)^3}{3}$

Hence $|a_n| \le \frac{(n+1)^3}{3n^4}$.

But $\lim_{n \to \infty} \frac{(n+1)^3}{3n^4} = \lim_{n \to \infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4} = \lim_{n \to \infty} \frac{\left(1 + 3\frac{1}{n} + 3\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^3\right)}{n}$

$$= st\left(\frac{1 + \frac{3}{H} + \frac{3}{H^2} + \frac{1}{H^3}}{H}\right) = st\left(\frac{\text{finite}}{\text{infinite}}\right) = 0.$$

Example. $\lim \frac{n!}{2^n}$

$$\frac{n!}{2^n} = \frac{1\cdot 2\cdot 3 \cdots n}{2\cdot 2\cdot 2 \cdots 2} = \frac{1}{2}\cdot\frac{2}{2}\underbrace{\left(\frac{3}{2}\cdot\frac{4}{2}\cdot\cdots\frac{n}{2}\right)}_{n-2}$$

Hence $\frac{n!}{2^n} \geq \frac{1}{2}\left(\frac{3}{2}\right)^{n-2}$, tends to infinity.

Hence $\lim \frac{n!}{2^n} = \infty$ by Sandwich theorem.

Example.
$$\lim_{n\to\infty} \frac{99n}{n^2-2n} = \lim_{n\to\infty} \frac{99}{n\left(1-\frac{2}{n}\right)} = \lim_{n\to\infty}\left(\frac{99}{1-\frac{2}{n}}\right)\lim_{n\to\infty}\left(\frac{1}{n}\right)$$

$$= st\left(\frac{99}{1-\underbrace{\left(\frac{2}{H}\right)}_{\text{infinitesimal}}}\right)\cdot 0 = 99\cdot 0 = 0.$$

Eliminate irrationality

$$\lim_{n\to\infty}\left(\sqrt[4]{n+1} - \sqrt[4]{n}\right) =$$

$x^4 - y^4 = (x-y)(x^3+x^2y+xy^2+y^3)$, or

Note $a - b = \left(a^{1/4} - b^{1/4}\right)\left(a^{3/4} + a^{2/4}b^{1/4} + a^{1/4}b^{2/4} + b^{3/4}\right)$

Therefore

$$a^{1/4} - b^{1/4} = \frac{a-b}{a^{3/4}+a^{2/4}b^{1/4}+a^{1/4}b^{2/4}+b^{3/4}}$$

Then
$$\lim_{n\to\infty}\left(\sqrt[4]{n+1}-\sqrt[4]{n}\right) = \frac{(n+1)-n}{(n+1)^{3/4}+(n+1)^{2/4}n^{1/4}+(n+1)^{1/4}n^{2/4}+n^{3/4}}$$

$$\leq \frac{1}{4n^{3/4}}$$

Now $\lim_{n\to\infty}\frac{1}{4n^{3/4}} = st\left(\frac{1}{4H^{3/4}}\right) = 0$
$\uparrow$ infinite

Hence by sandwich $\lim\left(\sqrt[4]{n+1}-\sqrt[4]{n}\right) = 0$

Example. $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$ This is e by definition. Now do $\left(\frac{n^2+1}{n^2}\right)^{n^2+n}$

6 1/2

26 nov

Integer part $[x]$.

Lemma. Let $p \in \mathbb{N}$. Then $n - p\left[\frac{n}{p}\right] = k$

where $k = 0, 1, \dots p-1$, and moreover $n \equiv k \pmod{p}$.

Proof. Let $n = mp + k$. Then $\left[\frac{n}{p}\right] = \left[\frac{mp+k}{p}\right] = \left[m + \frac{k}{p}\right] = m$.

Hence $p\left[\frac{n}{p}\right] = pm$ and $n - p\left[\frac{n}{p}\right] = mp + k - mp = k$.

Subsequence: e.g. $a_n = (-1)^n$ diverges

but $b_n = a_{2n}$ converges since

$b_n = (-1)^{2n} = 1$ and $\lim_{n\to\infty} b_n = 1$

Similarly $c_n = a_{2n+1}$ converges since

$c_n = (-1)^{2n+1} = -1$ $\lim_{n\to\infty} c_n = -1$

Example of subsequence. Let $a_n = \frac{1}{n}$. $(\to 0)$

Define subsequence $b_n = a_{n^2}$. Then $b_n = \frac{1}{n^2}$

converges to zero faster

Now let $f(n)$ be any (strictly increasing) $f: \mathbb{N} \to \mathbb{N}$

Given $\langle a_n : n \in \mathbb{N} \rangle$ define subsequence

$b_n = a_{f(n)}$

Definition Partial limit of $\langle a_n \rangle$ is a limit of a suitable subsequence.

Ex. Let $a_n = n - 5\left[\frac{n}{5}\right]$. Find all partial limits.

Let $n = 5m + k$, $k = 0, 1, 2, 3, 4$. Then $a_n = k$ by the lemma.

Subsequences $b_n = a_{5n}$ is identically zero hence $\lim b_n = 0$ similarly

$c_n = a_{5n+1}$ is identically $c_n = 1$ hence $\lim c_n = 1$ etc

$d_n = a_{5n+2}$ .. $\lim d_n = 2$.

etc. list of partial limits $\{0, 1, 2, 3, 4\}$.

Better notation: Let $s_n^{(k)} = a_{5n+k}$ where $k \in \{0, .., 4\}$.

Then $\lim_{k \to \infty} s_n^{(k)} = k$.

Example. $b_n = \dfrac{\sqrt[3]{\sqrt[3]{n} + 2\sqrt[4]{n}}}{\sqrt[5]{6n + \sqrt{7n}}}$ Find limit.

$$b_n = \frac{(n^{1/3} + 2n^{1/4})^{1/3}}{(6n + (7n)^{1/2})^{1/5}} = \frac{\left[n^{1/3}\left(1 + 2n^{-1/12}\right)\right]^{1/3}}{\left[n\left(6 + 7^{1/2}n^{-1/2}\right)\right]^{1/5}} =$$

$$= \frac{n^{1/9}\left(1 + \frac{2}{n^{\frac{1}{12}}}\right)^{1/3}}{n^{1/5}\left(6 + \frac{7^{1/2}}{n^{1/2}}\right)^{1/5}} = \frac{1}{n^{1/5 - 1/9}} \frac{\left(1 + \frac{2}{n^{1/12}}\right)^{1/3}}{\left(6 + \frac{\sqrt{7}}{n^{1/2}}\right)^{1/5}}$$

$$= \frac{1}{n^{\frac{4}{45}}} \frac{(1 + \ldots)^{1/3}}{(6 + \ldots)^{1/5}}.$$

Now $\lim b_n = st\left(\frac{1}{H^{4/45}}\right) \dfrac{st(1 + \ldots)^{1/3}}{st(6 + \ldots)^{1/5}}$

$$= st\left(\frac{1}{\text{infinite}}\right) \frac{\left(1 + st\left(\frac{2}{n^{1/12}}\right)\right)^{1/3}}{\left(6 + st\left(\frac{\sqrt{7}}{n^{1/2}}\right)\right)^{1/5}}$$

$= 0 \cdot \frac{1}{1} = 0$. Hence all partial limits are zero.

<u>Definition</u> $\overline{\lim}\, a_n = \limsup a_n =$

$= \max\{L : L$ is a partial limit of $a_n\}$.

$\underline{\lim} = \liminf$

$\overline{\lim}\,(-1)^n = 1$, $\underline{\lim}\,(-1)^n = -1$.

<u>Example</u>

$= 2^{((-1)^n)}$ Then $a_{2n} = 2$, $a_{2n+1} = \frac{1}{2}$

Example. Let $a_n =$

$\overline{\lim}\, a_n = 2$, $\underline{\lim}\, a_n = \frac{1}{2}$. (for $a_n > 0$).

<u>Note</u> $\overline{\lim}\left(\frac{1}{a_n}\right) = \frac{1}{\underline{\lim}\, a_n}$ then $\dfrac{\overline{\lim}}{\underline{\lim}} \geq 1$ with equality iff limit exists.

<u>Note</u> $\overline{\lim} \geq \underline{\lim}$ for $a_n > 0$.