MATHEMATICAL EDUCATION

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PEAKS, RIDGE, PASSES, VALLEY AND PITS
A Slide Study of $f(x,y)=Ax^2+By^2$

CLIFF LONG

The advent of computer graphics is making it possible to pay heed to the suggestion, "a picture is worth a thousand words." As students and teachers of mathematics we should become more aware of the variational approach to certain mathematical concepts, and consider presenting these concepts through a sequence of computer generated pictures.

To illustrate this notion, consider the following. In the study of functions of two variables it is usually emphasized that a regular non-planar point on a smooth surface can be classified as one of three distinct types: elliptic, parabolic, hyperbolic [1]. These may be illustrated using the functions $f(x,y)=Ax^2+By^2$ with the origin being:

(a) elliptic if $A\cdot B>0$;
(b) parabolic if $A\cdot B=0$, $A\neq B$ (planar if $A=B=0$);
(c) hyperbolic if $A\cdot B<0$.

The slides reproduced here (see page 371) were made at Bowling Green State University from the screen of an Owens-Illinois plasma panel which is an output device for a Data General Nova 800 mini-computer. Many slide sets and super eight movies have been produced by college mathematics teachers under an NSF grant for "Computer Graphics for Learning Mathematics." The institute was held at Carleton College in Northfield, Minnesota, 55057, during the summers of 1973 and 1974. It was under the direction of Dr. Roger B. Kirchner, who, with no small amount of personal effort, has made these slides and movies available at minimum reproduction cost.

It must of course be kept in mind that while one good picture may be worth a thousand words, a thousand poorly chosen ones may be worthless.

Reference

1. D. Hilbert and S. Cohn-Vossen, Geometry and the Imagination, Chelsea, New York, 1952.

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THE TEACHING OF ELEMENTARY CALCULUS USING THE NONSTANDARD ANALYSIS APPROACH

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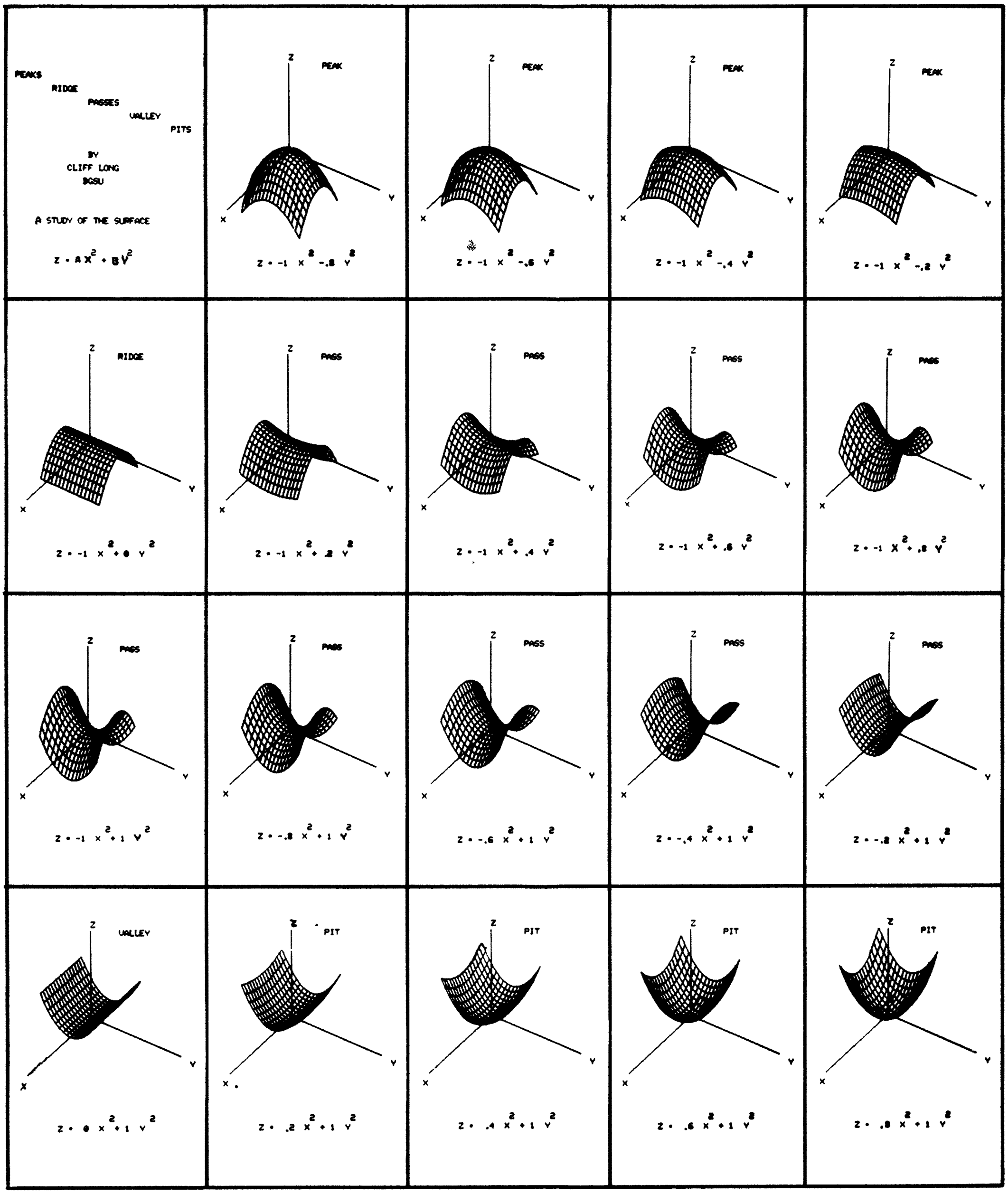
In the 1960's a mathematical logician, Abraham Robinson, found a way to make rigorous the intuitively attractive infinitesimal calculus of Newton and Leibniz, beginning a branch of mathematics called nonstandard analysis. When elementary calculus is developed from this nonstandard approach, the definitions of the basic concepts become simpler and the arguments more intuitive (see Robinson [2] or Keisler [1]). For example, the definition of the continuity of a function f at a point c is simply that x infinitely close to c implies that $f(x)$ is infinitely close to $f(c)$.



It would seem that there ought to be considerable pedagogical payoff from this greater simplicity and closeness to intuition. However, anyone considering using this approach will have questions that need to be answered. Will the students "buy" the idea of infinitely small? Will the instructor need to have a background in nonstandard analysis? Will the students acquire the basic calculus skills? Will they really understand the fundamental concepts any differently? How difficult will it be for them to make the transition into standard analysis courses if they want to study more mathematics? Is the nonstandard approach only suitable for gifted mathematics students?

In order to get answers to some of these questions, an experiment was carried out, using an institutional cyclic design (see Sullivan [3]). Five instructors who were teaching a traditional calculus course in schools located in the Chicago-Milwaukee area during the 1972–73 school year volunteered to use the nonstandard approach during the 1973–74 school year. The text used was H. Jerome Keisler's *Elementary Calculus: An Approach Using Infinitesimals*, published by Prindle, Weber & Schmidt, which is the first text book to adapt the ideas of Robinson to a first year calculus course. It seemed important that the same teachers be involved each year, teaching basically the same student population. Otherwise, differences in attitude and performance might well be written off to differences in instructors or in student populations. Only instructors who had taught calculus several times before were asked to participate in the study so that they would be in a good position to make comparisons.

Four of the five schools participating were small private colleges. The fifth was a public high school with over 2000 students in Glendale, an upper middle-class suburb of Milwaukee. At Saint Xavier College, the control and experimental classes consisted of students planning to major in mathematics; at Lake Forest College, of students in an honors program; at Nicolet High School, of a group of accelerated mathematics students in an advanced placement class. Barat College offers only one calculus course, and the same is true of Mount Mary College.

The assumption that the two groups would be comparable in ability was supported by a check of the SAT mathematics ability scores which were available, i.e., scores for 58 out of the 68 students in the control group and for 55 out of the 68 students in the experimental group. The distribution of scores is shown in the table below.

TABLE 1: *SAT Math Ability Scores*

	Control Group	Experimental Group
700+	13	13
600–699	29	29
500–599	14	11
400–499	2	2

The means chosen to collect data were the following: a calculus test given to both groups of students; interviews with the instructors who taught the control and experimental classes; and a questionnaire filled out by all those who had used Keisler's book within the past five years.

The calculus test was a fifty-minute exam which the instructors did not see until after it had been given to both groups. The purpose was to explore whether or not there did seem to be differences in performance between the two groups. The questions tested the ability of the students to define basic concepts, compute limits, produce proofs, and apply basic concepts.

The single question which brought out the greatest differences between the two groups was question 3:

$$\text{Define } f(x) \text{ by the rule } f(x) = x^2 \quad \text{for } x \neq 2,$$
$$f(x) = 0 \quad \text{for } x = 2.$$

Prove, using the definition of limit, that $\lim_{x\to 2} f(x) = 4$.

The responses of the students are summarized in Table 2.

The fact that there were a greater number of students in the experimental group willing to try to solve the problem was part of a pattern. Table 3 shows the numbers in the two groups who attempted an answer to the corresponding test questions.

There was also a notable difference between the two groups in the way in which they responded to question 6, which called for an explanation as to why a certain integral represented the volume of a given solid. In the control group there were just 9 students who spoke of the integral as a sum, compared with 16 in the experimental group. And of these students, only 3 in the first group, but 10 in the nonstandard group, saw it as a sum of small volumes rather than a sum of small areas.

TABLE 2: *Student Responses to Question* 3

	Control Group (68 students)	Experimental Group (68 students)
Did not attempt	22	4
Standard arguments		
satisfactory proof	2	
correct statements falling short of proof (e.g., one is only concerned with $x \neq 2$)	15	14
incorrect arguments	29	23
Nonstandard arguments		
satisfactory proof		25
incorrect arguments		2

TABLE 3: *Number of Students Attempting a Solution*

	Control Group (68 students)	Experimental Group (68 students)
Defining basic concepts	48	52
Computing limits	49	68
Producing proofs	18	45
Applying basic concepts	60	60

One of the instructors mentioned during an interview that formerly his students had found the use of the symbol dx in the integral very mysterious, but that the students in the experimental class had seemed completely satisfied. This observation was supported by the answers to question 6. The first year, only 2 students commented on the significance of dx. The second year, 16 students brought it into their explanations.

As for success in giving definitions, there was little difference between the two groups, except that the nonstandard group gave nonstandard definitions. (Since both standard and nonstandard definitions are presented in Keisler's book, a choice was involved.) The two groups also did about equally well in computing limits. The only striking difference was that 23 students in the standard group took derivatives in computing limits without there being any justification for doing so (including some students from each of the 5 schools). Only one student in the experimental group made this mistake.

Seeking to determine whether or not students really do perceive the basic concepts any differently is not simply a matter of tabulating how many students can formulate proper mathematical definitions. Most teachers would probably agree that this would be a very imperfect instrument for measuring understanding in a college freshman. But further light on this and other questions can be sought in the comments of the instructors. The instructor questionnaire was responded to by all of the 12 instructors who had taught a course using the text *Elementary Calculus: An Approach Using Infinitesimals* within the past three years. The responses are tabulated in Table 4, with the numbers indicating how many instructors gave each response.

Note that the group as a whole responded in a way favorable to the experimental method on every item in Part One (agreeing with the advantages cited, disagreeing with the objections proposed). Note also that in Part Two there was almost complete agreement that the proofs of nonstandard calculus are easier to explain and closer to intuition. This seems quite remarkable since the instructors would be much more familiar with standard proofs. The experimental group was also given the edge regarding the ease with which they were able to learn the basic concepts. One instructor commented that, "When my most recent class were presented with the epsilon-delta definition of limit, they were outraged by its obscurity compared to what they had learned."

In the individual interviews with the 5 teachers who had taught the control and experimental classes, 3 of them said they felt that the students in the experimental class had a much better feeling for limits. One of them found the explanation in the fact that a student can move forward in thought (if

TABLE 4: *Instructor Questionnaire*: *Part One*

(−2 strongly disagree, −1 disagree, but not strongly, 0 neither agree nor disagree, 1 agree, but nor strongly, 2 strongly agree)

The responses below refer to the experimental classes.

	−2	−1	0	1	2
1. The students had a problem with accepting axioms for the hyperreal numbers.	3	4	1	4	
2. The students seemed to find "infinitely small" a natural concept.		1	2	4	5
3. The time that must be used for nonstandard material makes it difficult to cover topics that ought to be included in a 1st year course.	8	2	1	1	
4. The course seemed to give the students a better feeling for the historical development of mathematics than a standard course.	1	3	3	4	1
5. I think that a student who has had two semesters of nonstandard calculus will be at a disadvantage in a standard 3rd semester calculus course.	6	4	1	1	
6. I enjoyed teaching calculus using this approach.			2	2	8
7. I probably should not have tried to teach the course without a better background in nonstandard analysis.	9	1	1	1	
8. I feel that the experience of teaching calculus from the infinitesimal approach will enrich my future teaching of calculus.			1	5	6
9. I am afraid that the introduction of infinitesimals left the students confused about the real numbers.	4	6		2	
10. I would prefer to use the nonstandard approach the next time I teach calculus.		2	1	4	5

Instructor Questionnaire: *Part Two*

(S Standard, NS Nonstandard, ND No Difference)

The instructors indicated which approach seemed to them to have the advantage or that neither approach seemed to have an advantage over the other.

	S	NS	ND
1. The students learn the basic concepts of calculus more easily.		8	4
2. The students seem to be more "turned on."		5	7
3. The proofs are easier to explain and closer to intuition.	1	10	1
4. The students find it easier to formulate their questions.		2	9
5. The students end up with a better understanding of the basic concepts of calculus.		5	7

I pick a point x which is infinitely close to c, $f(x)$ will be infinitely close to L) rather than backward (if I want $f(x)$ to be within epsilon of L, I must pick x to be within delta of c). It was also pointed out that the nonstandard approach makes it easier to illustrate concepts like the derivative on a static blackboard since one is dealing with a single number — a representative infinitesimal — and applying operations to it rather than letting something approach zero.

Two instructors expressed the opinion that the nonstandard method of learning calculus has special merit for students planning to major in engineering or physics, fields in which infinitesimals have always been considered a useful tool. On the other hand, some uncertainty was voiced on the question of how well students who want to study more analysis will be able to make the transition from an experimental class to a traditional course. Conversations with students at the University of Wisconsin, who had been in nonstandard calculus classes, suggest that the attitude of the instructor in the standard class may be the crucial factor. The students will have the necessary mathematical background to make the transition, but perhaps they ought to be prepared for the fact that their future instructors may or may not be convinced of this fact. As G. R. Blackley remarked in a letter to Prindle, Weber & Schmidt, concerning *Elementary Calculus: An Approach Using Infinitesimals*, "Such problems as might arise with the book will be political. It is revolutionary. Revolutions are seldom welcomed by the established party, although revolutionaries often are."

Certainly there is no claim to have found an instant solution to the problem of teaching calculus. This is not "Calculus made easy." Hard work and practice will still be required of the student, and a poor background in algebra will still make it difficult to learn calculus.

On the other hand, there does seem to be considerable evidence to support the thesis that this is indeed a viable alternate approach to teaching calculus. Any fears on the part of a would-be experimenter that students who learn calculus by way of infinitesimals will achieve less mastery of basic skills have surely been allayed. And it even appears highly probable that using the infinitesimal approach will make the calculus course a lot more fun both for the teachers and for the students.

References

1. H. J. Keisler, Elementary Calculus: An Approach Using Infinitesimals, Prindle, Weber & Schmidt, Boston, 1971.
2. A. Robinson, Nonstandard Analysis, North Holland, Amsterdam, 1970.
3. K. A. Sullivan, Doctoral Dissertation, University of Wisconsin, Madison, 1974.

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USE OF CANNED COMPUTER PROGRAMS IN FRESHMAN CALCULUS

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1. Introduction. Many articles and newsletters ([1], [2], [3], [4], and their references) have been published on the subject of "computer calculus," which means a calculus course in which the students write and run computer programs. It is the purpose of this article to describe a freshman calculus course that uses canned computer programs, to argue that canned programs are an excellent alternative to user-written programs, and to encourage instructors who have not used any computer programs in freshman calculus to try using canned ones.

2. General description. The following description refers to a course of 100 math, science and economics majors taught at Simmons College. Students were enrolled in 4 sections, each of which met 3 hours a week. The text was Thomas' *Calculus and Analytic Geometry* ([5]). Students had periodic access to two teletype terminals hooked into a Hewlett Packard HP-2000C. The terminal usage was a required part of the course; computer exercises were collected, but not graded. Students, working in pairs, ran 6 programs each semester, for a total expenditure of $1\frac{1}{2}$ hours or \$7.50 per student per semester.

3. The logistics of terminal usage. One-third of the way through the fall semester, the calculus class was shown how to use the terminal. Students did the actual typing of responses to the first