תרגול כיתה 10

The integration problem can be expressed in a slightly more general way by introducing a positive weight function ω into the integrand, and allowing an interval other than [−1, 1]. That is, the problem is to calculate

$$\int_a^b \omega(x) f(x) dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

for some choices of a, b, and ω are given for Abramowitz and Stegun (A & S).

Interval	**ω(*x*)**	**Orthogonal polynomials**	**A & S**
[−1, 1]	1	Legendre polynomials	25.4.29
(−1, 1)	$(1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha, \beta > -1$	Jacobi polynomials	25.4.33 (β = 0)
(−1, 1)	$\frac{1}{\sqrt{1-x^2}}$	Chebyshev polynomials (first kind)	25.4.38
[−1, 1]	$\sqrt{1-x^2}$	Chebyshev polynomials (second kind)	25.4.40
[0, ∞)	e^{-x}	Laguerre polynomials	25.4.45
(−∞, ∞)	e^{-x^2}	Hermite polynomials	25.4.46

The polynomial $p_n(x)$ is said to be an *orthogonal polynomial* of degree n associated to the weight function $\omega(x)$ *if*

$$\int_a^b \omega(x) p_n(x) p_m(x) dx = 0, \quad n \neq m$$

$p_n(x)$ is unique up to a constant normalization factor, moreover any *orthogonal polynomial* p_n of degree n fulfill

$$\int_a^b \omega(x)\, x^k p_n(x)\, dx = 0, \quad \text{for all } k = 0, 1, \ldots, n-1.$$

Fundamental theorem

If we pick the nodes to be the zeros of p_n, then there exist weights w_i which make the computed integral exact for all polynomials of degree $2n - 1$ or less.

Some low-order rules for solving the integration problem are listed below.

Number of points, n	Points, x_i	Weights, w_i
1	0	2
2	$\pm\sqrt{1/3}$	1
3	0	$8/9$
	$\pm\sqrt{3/5}$	$5/9$
4	$\pm\sqrt{\left(3-2\sqrt{6/5}\right)/7}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\left(3+2\sqrt{6/5}\right)/7}$	$\frac{18-\sqrt{30}}{36}$
5	0	$128/225$
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{10/7}}$	$\frac{322+13\sqrt{70}}{900}$
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{10/7}}$	$\frac{322-13\sqrt{70}}{900}$

```
s1=sqrt((3-2*sqrt(6/5))/7);
s2=sqrt((3+2*sqrt(6/5))/7); x=[-s2 -s1 s1 s2]

w1=(18+sqrt(30))/36; w2=(18-sqrt(30))/36;
w=[w2 w1 w1 w2]

syms z; for k=1:8
 intgr(k)=eval(int(z^k,z,-1,1)-dot(w,x.^k));
end
intgr

intgr =

  -0.0000        0        0        0        0        0        0   0.0116
```

The **Newton–Cotes Formula** of degree n is stated as

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} w_i\, f(x_i)$$

where $x_i = h\, i + x_0$, with h (called the *step size*) equal to $(x_n - x_0) \;/\; n = (b - a) \;/\; n$.

The w_i are called *weights*.

Degree	Common name	Formula	Error term
1	Rectangle rule, or midpoint rule	$(b-a)f_1$	$\frac{(b-a)^3}{24}\, f^{(2)}(\xi)$
1	Trapezoid rule	$\frac{b-a}{2}(f_0 + f_1)$	$-\frac{(b-a)^3}{12}\, f^{(2)}(\xi)$
2	Simpson's rule	$\frac{b-a}{6}(f_0 + 4f_1 + f_2)$	$-\frac{(b-a)^5}{2880}\, f^{(4)}(\xi)$
3	Simpson's 3/8 rule	$\frac{b-a}{8}(f_0 + 3f_1 + 3f_2 + f_3)$	$-\frac{(b-a)^5}{6480}\, f^{(4)}(\xi)$
4	Boole's rule	$\frac{b-a}{90}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4)$	$-\frac{(b-a)^7}{1935360}\, f^{(6)}(\xi)$

Composite trapezoidal rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left[\frac{f(a)+f(b)}{2}+\sum_{k=1}^{n-1} f\left(a+k\frac{b-a}{n}\right)\right].$$

This can alternatively be written as:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right)$$

where

$$x_k = a+k\frac{b-a}{n},\ \text{for } k=0,1,\dots,n$$

(one can also use a non-uniform grid).

The error of the composite trapezoidal rule is the difference between the value of the integral and the numerical result:

$$\text{error} = \int_a^b f(x)\,dx - \frac{b-a}{n}\left[\frac{f(a)+f(b)}{2}+\sum_{k=1}^{n-1} f\left(a+k\frac{b-a}{n}\right)\right].$$

This error can be written as

$$\text{error} = -\frac{(b-a)^3}{12n^2}f''(\xi),$$

where ξ is some number between a and b.

Composite Simpson's rule

Suppose that the interval $[a,b]$ is split up in n subintervals, with n an even number. Then, the composite Simpson's rule is given by

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0)+2\sum_{j=1}^{n/2-1} f(x_{2j})+4\sum_{j=1}^{n/2} f(x_{2j-1})+f(x_n)\right],$$

where $x_j = a + jh$ for $j = 0,1,...,n-1,n$ with $h = (b-a)/n$; in particular, $x_0 = a$ and $x_n = b$. The above formula can also be written as

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+4f(x_{n-1})+f(x_n)\right].$$

The error committed by the composite Simpson's rule is bounded (in absolute value) by

$$\frac{h^4}{180}(b-a) \max_{\xi \in [a,b]} |f^{(4)}(\xi)|,$$

where h is the "step length", given by $h = (b - a) / n$.